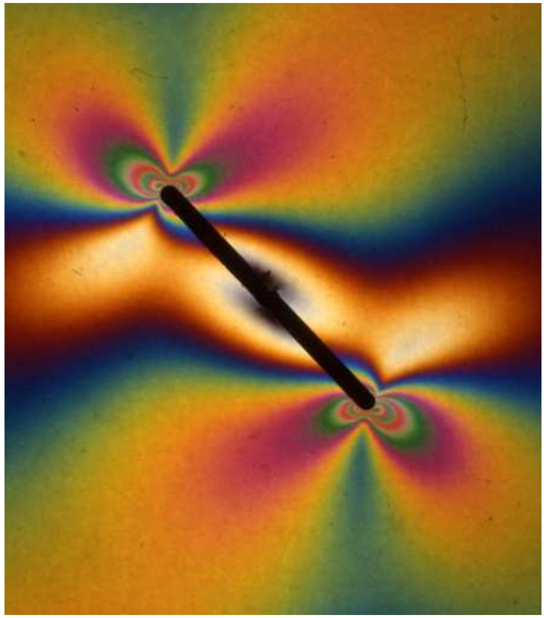
A new method to solve crack problems based on G2 theory

Elementary Fracture Mechanics



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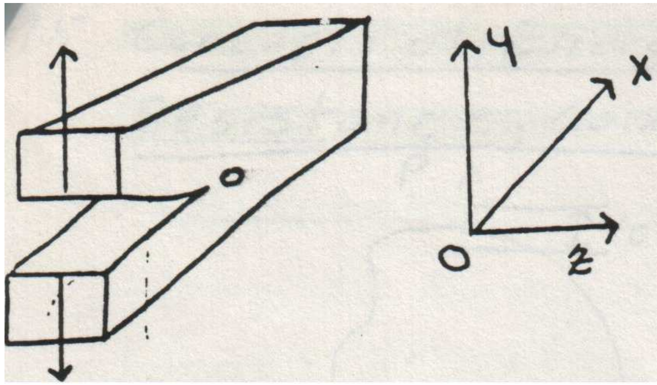
Dedicated to the memory of
Prof. Ioannis Vardoulakis

Historical notes

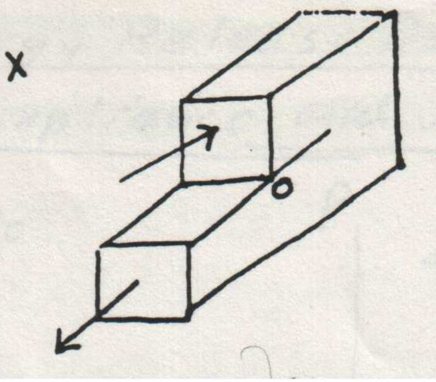
The presence of crack-like voids is known to have a profound effect on the strength and mechanical properties of materials. Griffith (1921) first showed that the low tensile strength of glass could be explained by the presence of slit like cracks. Irwin (1957) extended Griffith's ideas by introducing the concept that a Critical Energy Release Rate (ERR) governs fracture, while Barrenblatt (1962) developed the concepts of modes I, II, and III crack tip Stress Intensity Factors (SIF’s) as governing extension, shearing, and tearing modes of deformation. Irwin (1957) showed the equivalence of the ERR and the crack tip SIF’s, and crack tip SIF’s for a wide variety of crack geometries have long before been compiled (e.g. Sih (1973)).

Grittith A. A. The phenomenon of rupture and flow in solids. *Phil. Trans. R. Soc. Lond,* 221A, 163-198 (1921).
Irwin G. R. Analysis of stresses and strains near the ends of a crack traversing a plate. *J. Appl. Mech. 24,* 361-364 (1957).
Barrenblatt G. I. Mathematical theory of equilibrium cracks in **brittle** fracture. *Advances in Applied Mechanics,* Vol. 7. Academic Press, New York (1962).
Sih G. C. *Handbook of Stress Intensity Factors.* Institute of Fracture and Solid Mechanics, Lehigh University, Bethlehem (1973).

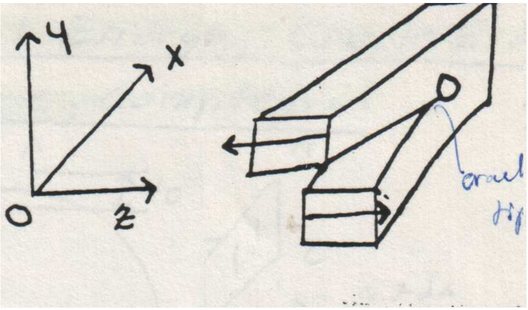
Modes of fracture (Ch 1 CWS)



$u_x = u_z = 0$ on the crack surface
Open mode Tension
symmetric wrt xy & xz planes
Mode I



$u_y = u_z = 0$ on the crack surface
Edge sliding In plane shear
symmetric wrt xy & skew-symmetric wrt xz plane
Mode II



$u_x = u_y = 0$ on the crack surface
Edge tearing
skew-symmetric wrt xy & xz planes
Mode III

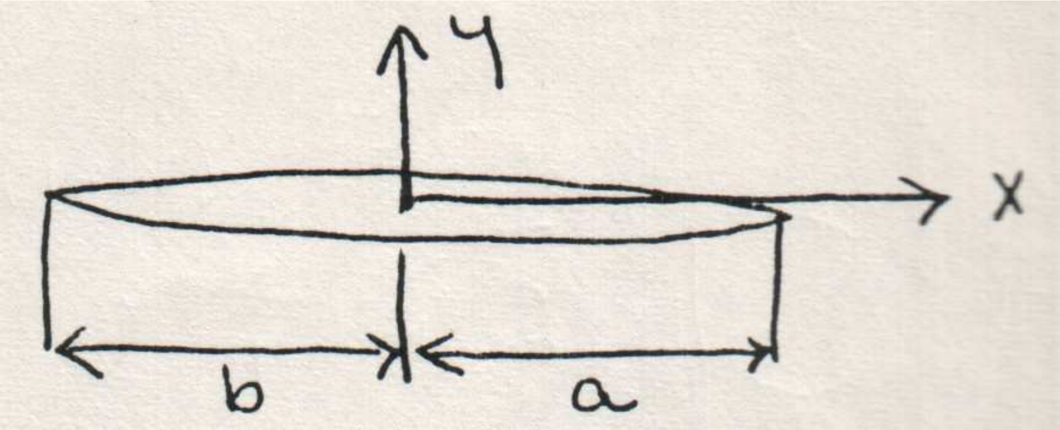
Mode II and III cracks bear a certain analogy to **edge and screw dislocations**, respectively, in the sense that **displacement discontinuities** exist along the crack surface behind the crack tips.

The superposition of the three basic modes is sufficient to describe the most general plane case of local crack tip stress and deformation fields.

Remarks:

1. The theoretical foundation of Fracture Mechanics may be soundly based upon the **Linear Small Strain Theory of Elasticity**, regardless of the phenomena occuring within the plastic zone at the crack tip which precipitates fracture.
2. The **failure criterion** may be based upon a limiting intensity of the local elastic stress field in the neighborhood of the crack tip. **This limit may be specified in terms of a single parameter.** This parameter was initially recognized historically to be the so called Strain Energy Release Rate (SERR) and was later shown to be related to the SIF (fracture toughness).
3. The **correct value of the SIF** to be applied in design should be based upon appropriate experimental data in combination with the **appropriate theoretical Stress analysis** (the latter is also our topic for discussion!!).
4. Cracks are taken to be branch cuts i.e. lines or surfaces of displacement discontinuity in 2D or 3D.

Stresses & displacements in cracked bodies (C



$$\left.\begin{aligned}\sigma_{yy} &= \mathrm{Re}\,Z_I + y\,\mathrm{Im}\,Z_I' \\ \tau_{xy} &= -y\,\mathrm{Re}\,Z_I'\end{aligned}\right\}$$

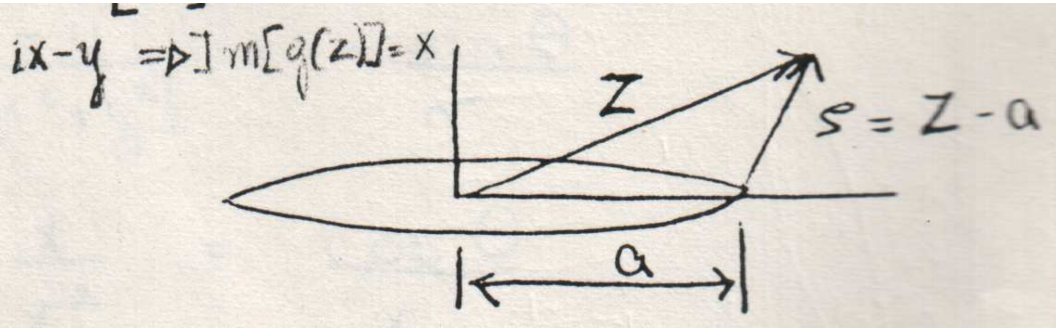
Step 1: Choose the following Westergaard function

$$z = x + iy$$

$$\boxed{Z_I = \frac{g(z)}{[(z+b)(z-a)]^{\frac{1}{2}}}}$$

Step 2: Moving the origin to the crack tip

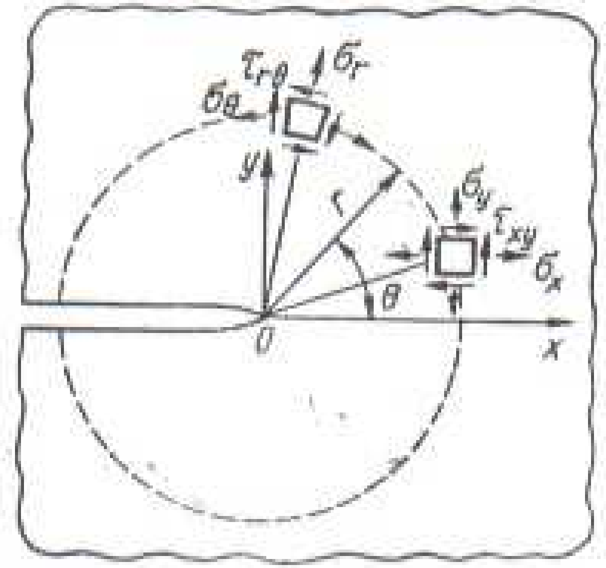
$$\zeta = z - \alpha$$



$$\boxed{Z_I = \frac{g(\zeta + a)}{[(\zeta + b + a)\zeta]^{\frac{1}{2}}} = \frac{f(\zeta)}{\zeta^{1/2}}}$$

Step 3: Expand f(ζ) in MacLaurin series (analytic): $f(\zeta) = \frac{K_I}{\sqrt{2\pi}} + a_1\zeta + a_2\zeta^2 + ...$

State of affairs at the crack t



$$\boxed{K_I = \lim_{|\zeta| \to 0} (2\pi\zeta)^{1/2} Z_I}$$

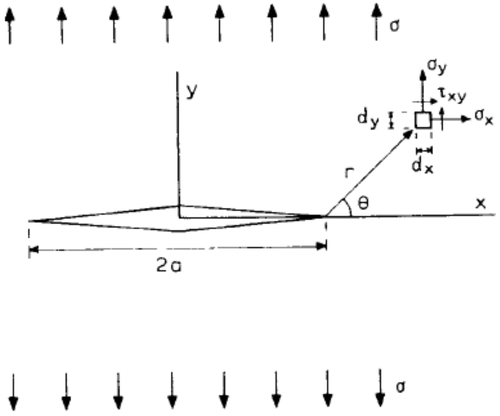


Figure 1.4. Crack in an infinite plate

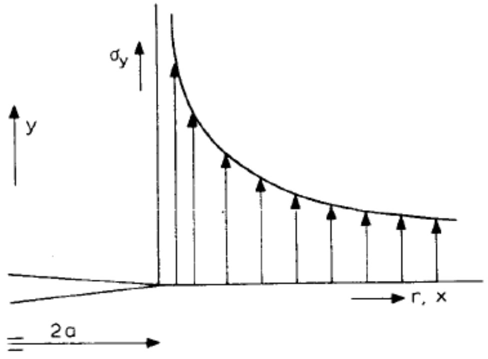


Figure 1.5. Elastic stress σ_y at the crack tip

Plane strain solution:

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r^{1/2}}} \cos\frac{\theta}{2}\left[1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r^{1/2}}} \cos\frac{\theta}{2}\left[1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right]$$

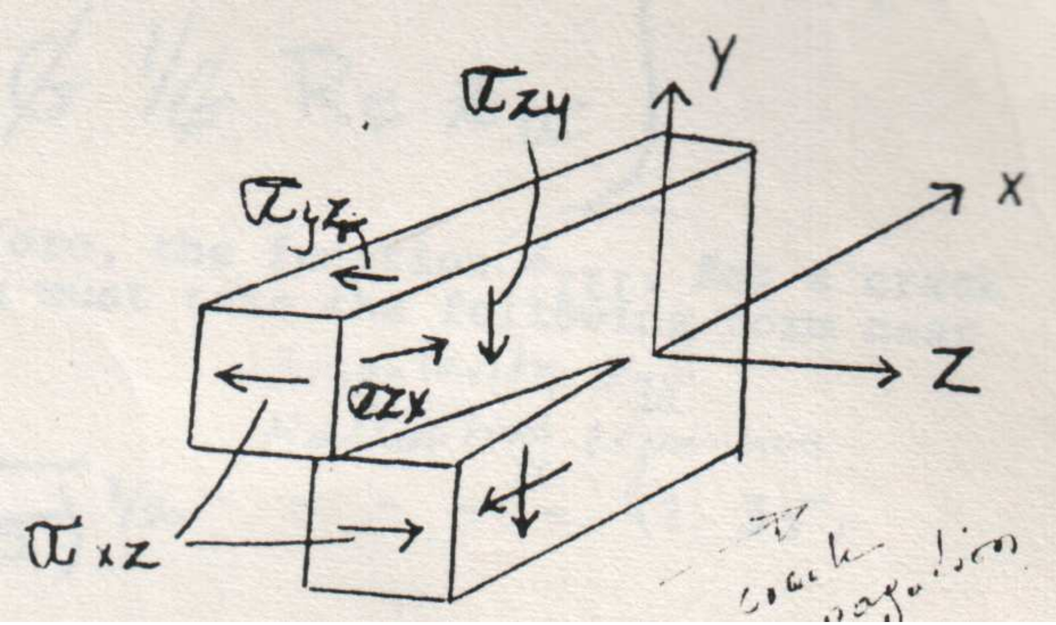
$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r^{1/2}}} \cos\frac{\theta}{2}\sin\frac{\theta}{2}\cos\frac{3\theta}{2}$$

$$\sigma_z = \nu\left(\sigma_x + \sigma_y\right)$$

$$u_x = \frac{K_I}{G}\left(\frac{r}{2\pi}\right)^{1/2} \cos\left(\frac{\theta}{2}\right)\left[1 - 2\nu + \sin^2\left(\frac{\theta}{2}\right)\right]$$

$$u_y = \frac{K_I}{G}\left(\frac{r}{2\pi}\right)^{1/2} \sin\left(\frac{\theta}{2}\right)\left[2(1-\nu) - \cos^2\left(\frac{\theta}{2}\right)\right]$$

$$u_z = 0$$



$$\left.\begin{aligned} u_x &= 0 \\ u_y &= 0 \\ u_z &= u_z(x,y) \end{aligned}\right\}$$

Kinematics of mode III crack

$$Z_{III}\Big|_{|\zeta|\to 0} = \frac{K_{1II}}{\sqrt{2\pi\zeta}}$$

Westergaard Stress function

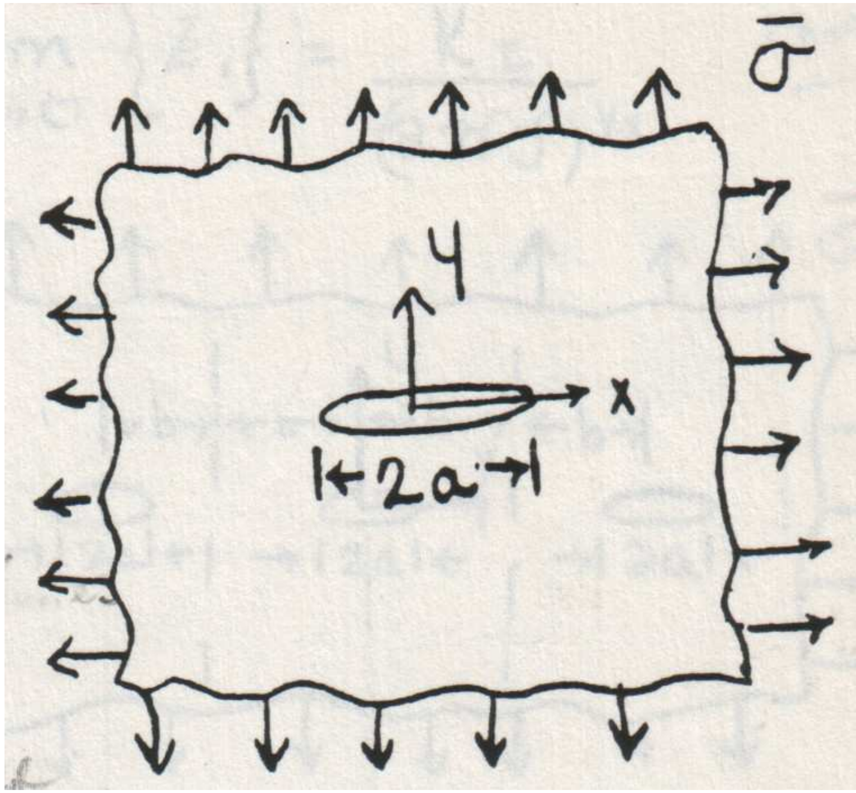
$$\left.\begin{aligned} \tau_{xz} &= \frac{-K_{III}}{(2\pi r)^{1/2}}\sin\frac{\theta}{2} \\ \tau_{yz} &= \frac{K_{III}}{(2\pi r)^{1/2}}\cos\frac{\theta}{2} \\ u_z &= \frac{K_{III}}{G}\left(\frac{2r}{\pi}\right)^{1/2}\sin\frac{\theta}{2} \end{aligned}\right\}$$

Solution

Determination of Stress Intensity Factors (Ch

1. Crack tip solutions using the Westergaard Stress Function

(Westergaard, 1936)



$$\sigma_{xx} = \operatorname{Re} Z_I - y \operatorname{Im} Z_I'$$

$$\sigma_{yy} = \operatorname{Re} Z_I + y \operatorname{Im} Z_I'$$

$$\tau_{xy} = -y \operatorname{Re} Z_I'$$

Choose: $Z_I = \dfrac{Cz}{(z^2 - a^2)^{1/2}}$

$$Z_I' = \frac{dZ_I}{dz} = \frac{C}{(z^2 - a^2)^{1/2}} - \frac{Cz^2}{\left(z^2 - a^2\right)^{3/2}} = -\frac{Ca^2}{\left(z^2 - a^2\right)^{3/2}}$$

The value of constant C may be determined from the BC's away from the crack and the resulting stresses must satisfy BC's at the crack surface.
At infinity:

$$\boxed{Z_I\Big|_{z\to\infty} = \lim \frac{Cz}{\left[1-\left(\dfrac{a}{z}\right)^2\right]^{1/2}} = C = C_1 + iC_2}$$

$$Z_I'\Big|_{z\to\infty} = \lim \frac{Ca^2}{\left[z^2-a^2\right]^{3/2}} = 0$$

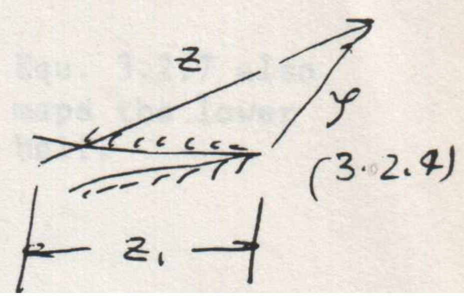
But the stresses

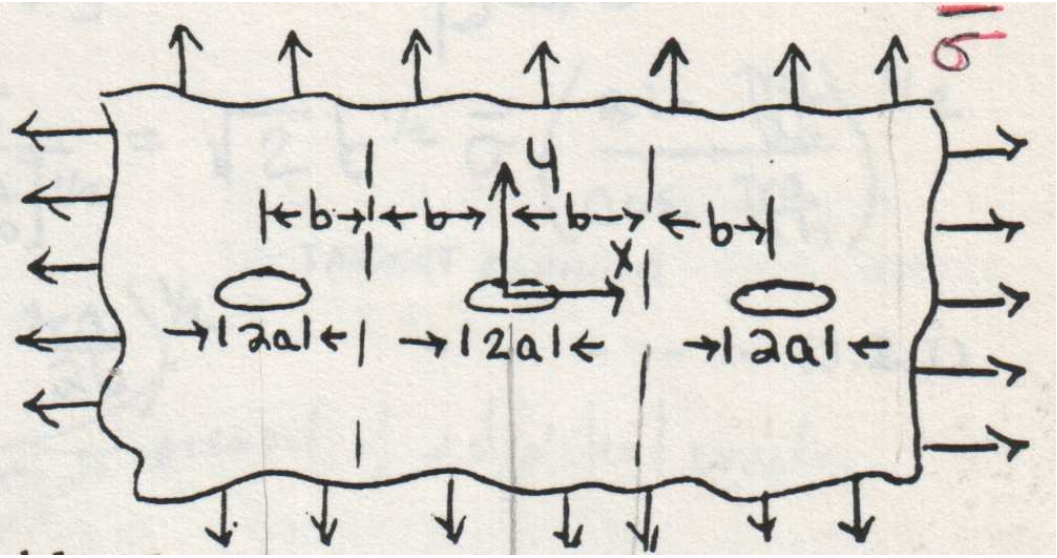
$$\sigma_{xx}\Big|_{z\to\infty} = C_1 = \bar{\sigma};\ \ \sigma_{yy}\Big|_{z\to\infty} = C_1 = \bar{\sigma};\ \ \tau_{xy}\Big|_{z\to\infty} = 0$$

Also note that along y=0 and in the segment –α<x<α the quantity $\left(x^2-a^2\right)^{1/2}$ is imaginary. But $\operatorname{Re} Z_{I=0}$ along the crack surface, hence $C_2 = 0$

Finally, by shifting the origin to a crack tip $\zeta = z - a = re^{i\theta}$, expanding Z_I about the crack tip in a Maclaurin series we obtain

$$\boxed{K_I = \lim_{|\zeta|\to 0} (2\pi\zeta)^{1/2} Z_I = \bar{\sigma}\sqrt{\pi\alpha}}$$





The stress function that solves this problem has been provided by Westergaard as follows

$$Z_I = \frac{\bar{\sigma}\sin\left(\frac{\pi z}{2b}\right)}{\left[\left(\sin\left\{\frac{\pi z}{2b}\right\}\right)^2 - \left(\sin\left\{\frac{\pi a}{2b}\right\}\right)^2\right]^{1/2}}$$

$$K_I = \lim_{|\zeta|\to 0}(2\pi\zeta)^{1/2} Z_I = \bar{\sigma}(\pi a)^{1/2}\left(\frac{2b}{\pi a}\tan\frac{\pi a}{2b}\right)^{1/2},$$

$$K_{II} = K_{III} = 0$$

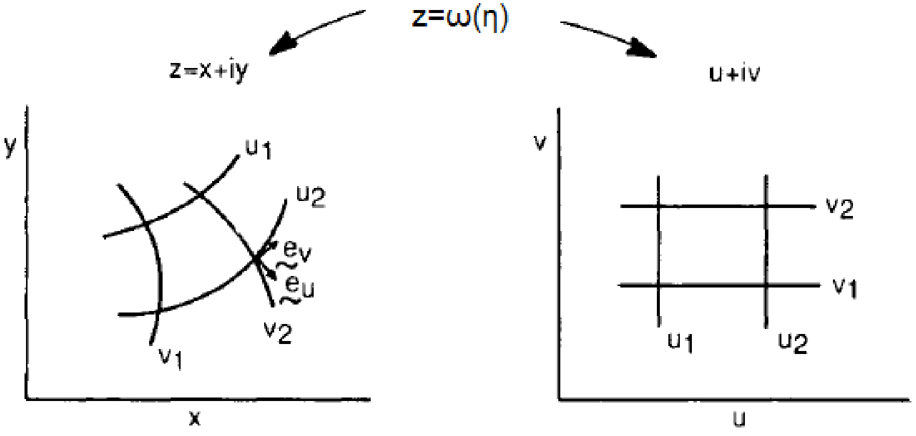
2a. Muskhelishvili’s approach using complex potential functions & conformal mapping

The concept of complex SIF

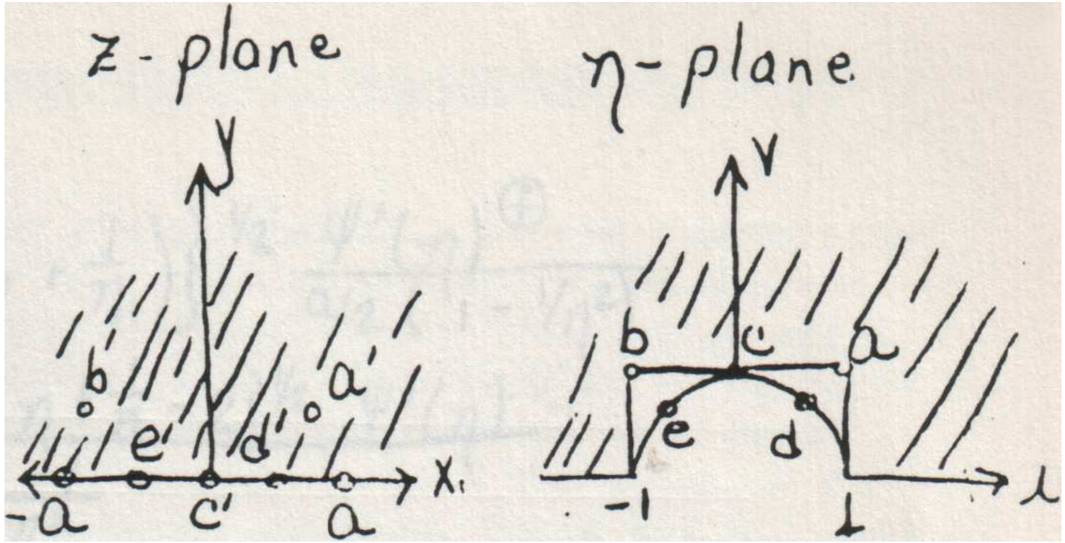
$$K = K_I - iK_{II} = 2\sqrt{2\pi}\lim_{z\to z_1}\phi'(z-z_1)^{1/2}$$

$$\boxed{\text{Consider the mapping function: } z=\omega(\eta)}$$

$$K = K_1 - iK_2 = 2\sqrt{2\pi}\mathop{Lim}_{\eta\to\eta_1}(\omega(\eta)-\omega(\eta_1))^{1/2}\left\{\frac{\phi'(\eta)}{\omega'(\eta)}\right\}$$



$$z = \frac{\alpha}{2}\left\{\eta+\frac{1}{\eta}\right\} \longrightarrow$$

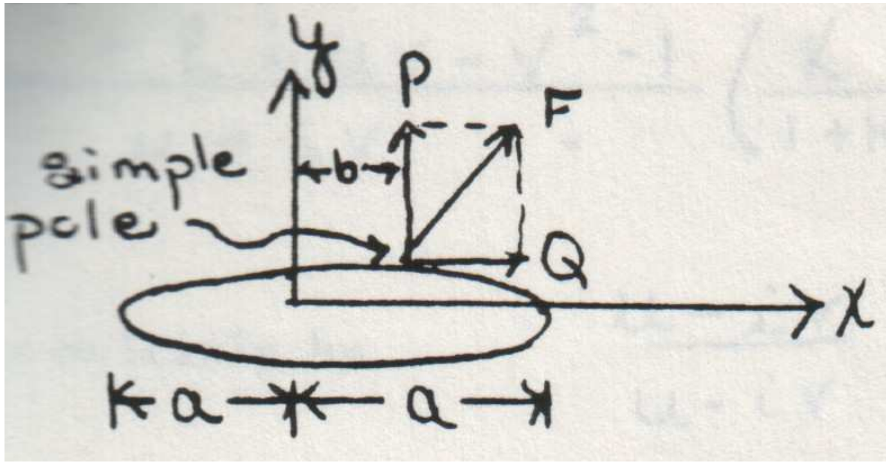


For the right-hand crack tip

$$K = 2\sqrt{\frac{\pi}{\alpha}}\phi'(1)$$

Example: Straight crack subjected to a concentrated force

This problem is of particular interest as it may be used as a Green's Influence Function to form the solution to other problems.



$$\phi(\eta) = \frac{F\alpha}{4\pi\left(\alpha^2 - b^2\right)^{1/2}}\left\{-\frac{1}{\eta} + \left(\frac{\eta_0}{\eta_0 - \eta}\right)\left[\left(\eta + \frac{1}{\eta}\right) - \left(\eta_0 + \frac{1}{\eta_0}\right)\right] + \left(\eta_0 + \frac{1}{\eta_0}\right)\left[\frac{\kappa}{1+\kappa}\log\eta - \log(\eta_0 - \eta)\right]\right\}$$

where: $F = P - iQ$

η_0 corresponds to z=b

$\kappa = 3 - 4\nu$

2b. Muskhelishvili’s approach using Cauchy type SIEs

Another efficient method for solving plane elasticity crack problems and estimating the SIFs at crack tips is the method which reduces the problem to a Cauchy type singular integral equation by considering a curvilinear crack composed of a series of edge dislocations.

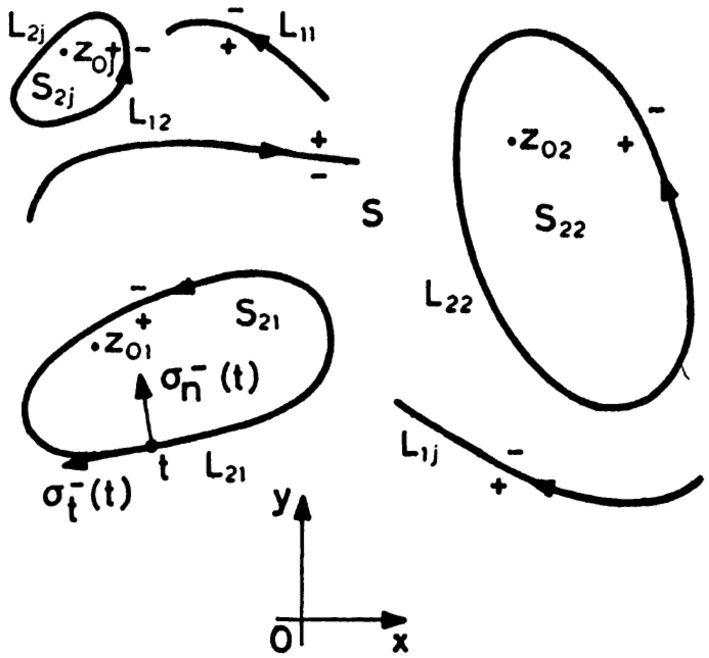


Fig. 1. An infinite plane medium with a set of holes and cracks.

complex potentials $\Phi(z)$ and $\Psi(z)$ in terms of Cauchy type integrals :

$$\Phi(z)=\phi'(z)=\frac{d\phi(z)}{dz},$$
$$\Psi(z)=\psi'(z)=\frac{d\psi(z)}{dz},$$

$$\Phi(z) = \frac{1}{2\pi i}\int_L \frac{\varphi(\tau)}{\tau-z}d\tau+\Gamma \ ,$$

$$\Psi(z) = -\frac{1}{2\pi i}\int_L \frac{\overline{\varphi(\tau)}-2\overline{q(\tau)}}{\tau-z}\,d\bar{\tau} - \frac{1}{2\pi i}\int_L \frac{\bar{\tau}\varphi(\tau)}{(\tau-z)^2}\,d\tau+\Gamma' \ ,$$

where L denotes both the cracks L_{1j} and the boundaries L_{2j} of the holes. In these equations the density $\varphi(t)$ is an unknown function of the points t of L, whereas the function $q(t)$ is defined by :

$$2q(t) = \left[\sigma_n^+(t)+i\sigma_t^+(t)\right] - \left[\sigma_n^-(t)+i\sigma_t^-(t)\right] \ ,$$

$$2q(t) = -\left[\sigma_n^-(t)+i\sigma_t^-(t)\right]$$

along the cracks L_{1j} and the boundaries L_{2j} of the holes respectively.

on the basis of the Plemelj formulae that the boundary conditions along L are :

$$2\mathrm{Re}\Phi^{\pm}(t) + \frac{d\bar{t}}{dt}\left[\bar{t}\Phi'^{\pm}(t)+\Psi^{\pm}(t)\right] = \sigma_n^{\pm}(t)-i\sigma_t^{\pm}(t) \ , \quad \frac{d\bar{t}}{dt} \equiv \frac{d\bar{t}/ds}{dt/ds} \ ,$$

where s is a variable denoting the arc-length.

3. Finite element methods

Discretization techniques such as FEM are not basically well suited to problems containing singularities. In order to deal with this difficulty, two basic approaches have been developed:

a) Non-singular crack-tip models
b) Singular crack-tip elements

a)<u>Non-Singular Models</u>

By using a very high density of elements near the tips, SIFs were derived from near tip Eqs. for stresses or displacements (latter more accurate than the former)

$$K_1 = (2\pi r)^{1/2} \lim_{r \to 0} \sigma_y$$

$$\boxed{K_1 = \lim_{r \to 0} \frac{E u_y}{4(1-\nu)^2} \left(\frac{2\pi}{r} \right)^{1/2}}$$

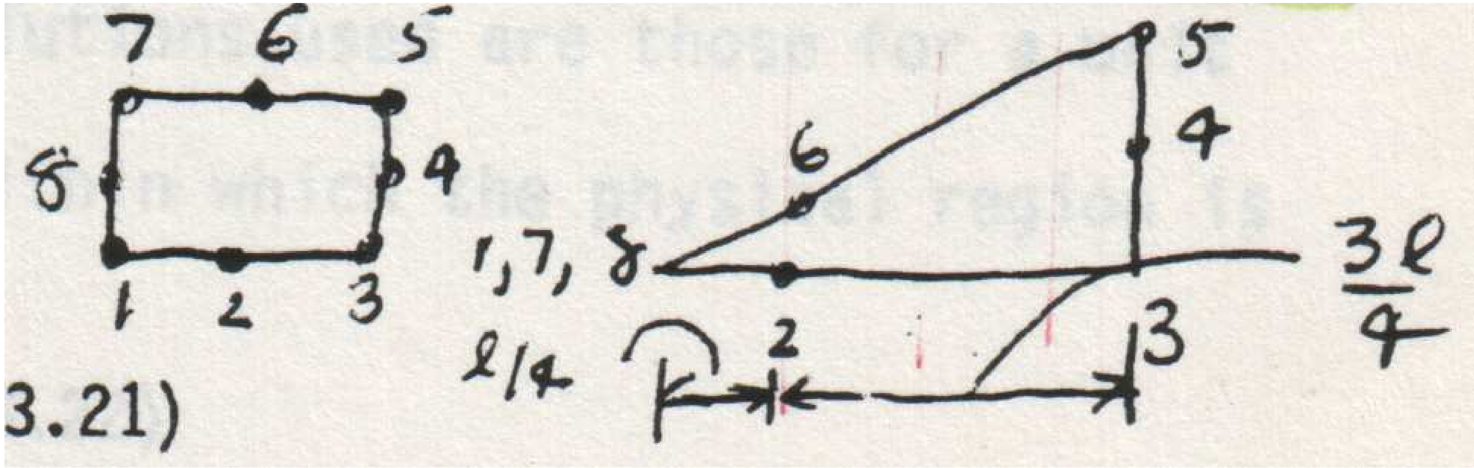
In order to reduce the requirement for so many near tip elements, several alternative approaches have been developed:

1. Compute U for 2 adjacent node crack lengths and calculate $g = \frac{\partial U}{\partial \alpha}$ or by $g = \lim_{a \to 0} \int \sigma_y dx$ (crack closure)
2. Use J contour integral
3. Compute $\frac{\partial C}{\partial \alpha}$ where C is crack compliance

b) Singular elements

An alternate approach involves the introduction of special crack tip singularity elements. We shall consider 2 classes of such elements:

1. Those with near tip stress and displacement fields described in terms of Westergaard, Muskhelishvili or William stress functions. This leads to a direct expression for K.
2. Elements which are modified by moving some nodes to the quarter points as shown below introduces a $r^{-1/2}$ singularity at the crack tip called isoparametric elements.



Motivation

It has been briefly shown that even though much achievement has been made in crack modeling techniques (analytical and numerical), a simple and practical crack modeling technique is still needed, in particular for **complex multiple crack growth problems (in Structural Geology, Rock Mechanics, Structural Engineering, Petroleum Engineering, Biomechanics etc).**

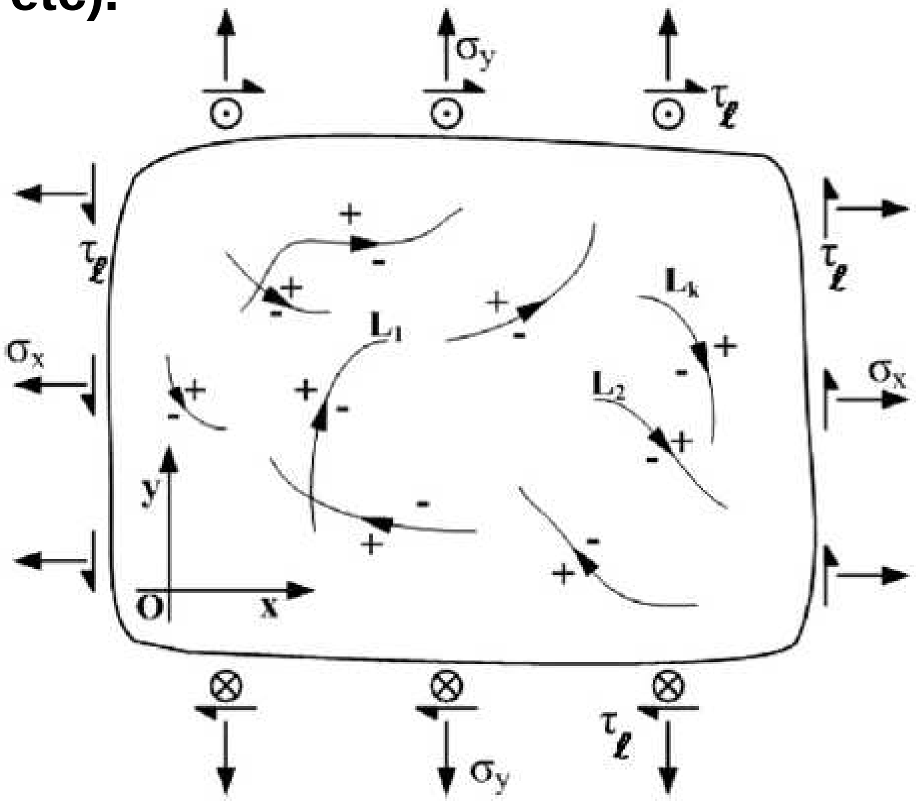


Fig. 1. A plane isotropic elastic body containing isolated or mutually intersecting cracks $L_1, L_2, \ldots, L_k$ and subjected to normal stresses σ_x, σ_y and shear stress τ_ℓ at infinity.

Constant normal and shear displacement discontinuities

The Displacement Discontinuity (DD) method, as originally presented by Crouch [1976,1990], is based on a solution of the **Neuber-Papkovitch displacement functions**, which expresses the stresses and displacements at a point due to a **Constant Displacement Discontinuity (CDD)** (dislocation) over a finite line segment (i.e. a branch cut).

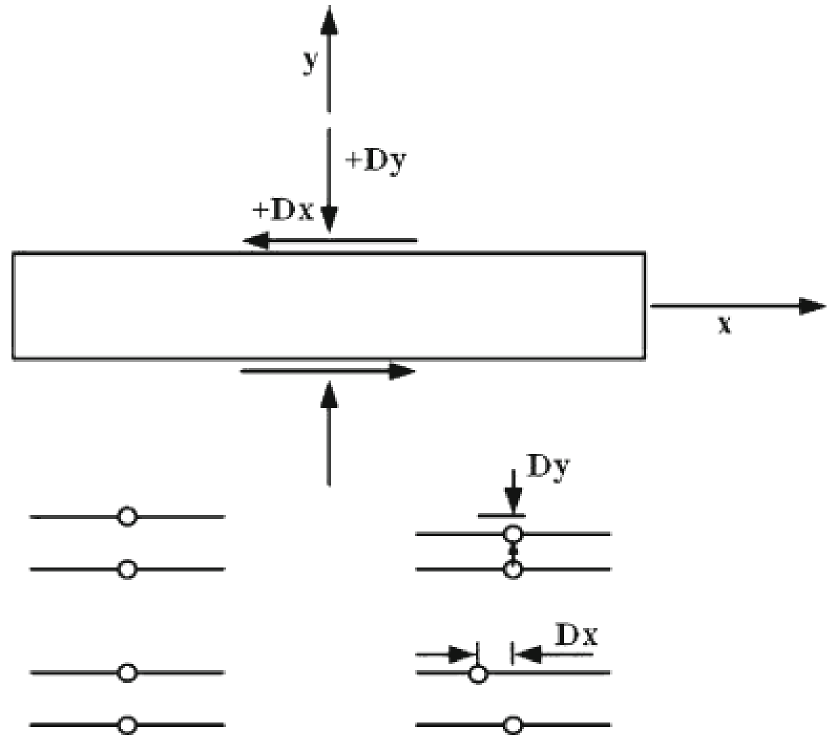
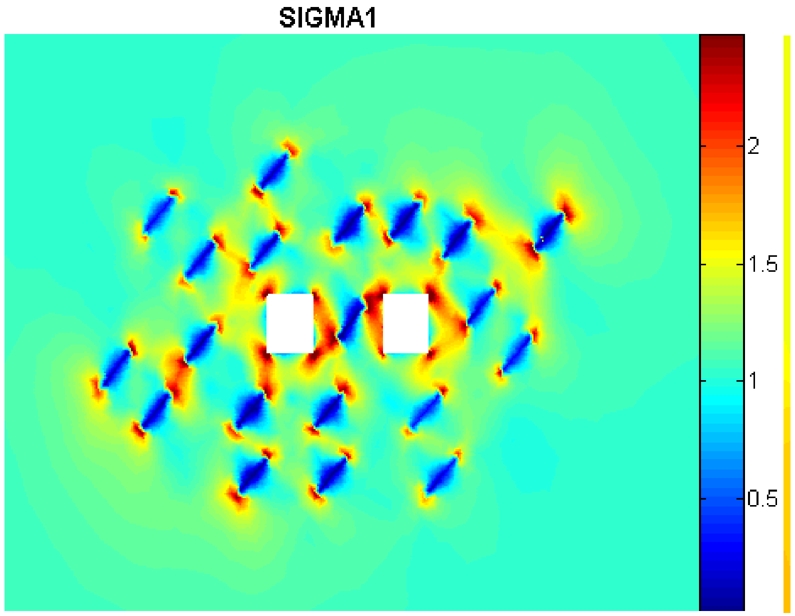
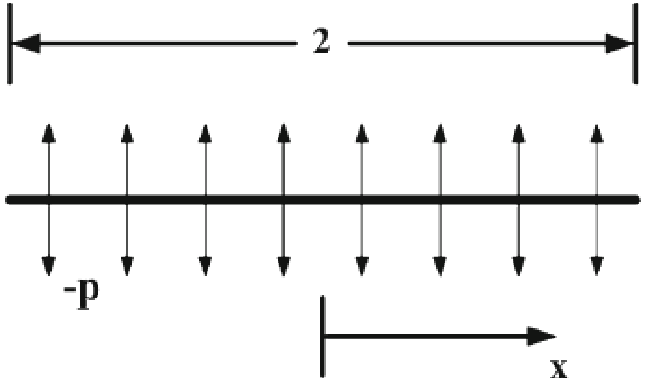


Fig. 1 Constant normal D_y and shear D_x displacement discontinuities along finite straight branch cuts

$$
\begin{aligned}
D_n &= u_n^- - u_n^+ \\
D_s &= u_s^- - u_s^+
\end{aligned}
$$



The uniformly pressurized crack problem



The uniformly pressurized crack problem

Analytical solution: $\hat{u}_y(x) = u_y(x,0^-) - u_y(x,0^+) = -\frac{2(1-\nu)}{G} p\left(1-x^2\right)^{1/2}$

Assumption: line segments are small enough so that the DD along Oy-axis can be taken as constant over each segment

Numerical approximation: $D_{y_j} \ , j = 1,\ldots,N$

Fundamental sol.: On each segment-j with a CDD along its entire length the stress is:

$$\sigma_{yy}(x,0) = -\frac{G}{\pi(1-\nu)} D_y \frac{1}{x^2-1}$$

If this constant DD occurs at the j-th element of the crack then eq. (3) takes the form

$$\sigma_{yy}(x,0) = -\frac{a_j G}{\pi(1-\nu)} D_{y_j} \frac{1}{(x-x_j)^2 - a_j^2}$$

The stress at the midpoint of the i-th segment due to a DD at the j-th segment is found by setting $x = x_i$

$$\sigma_{yy}(x_i,0) = \sigma_{yy_i} = -\frac{a_j G}{\pi(1-\nu)} D_{y_j} \frac{1}{(x_i-x_j)^2 - a_j^2}$$

,

Unknown (?)

Superposition: $$\boxed{\sigma_{yy}(x_i,0) = \sigma_{yy_i} = \sum_{j=1}^{N} A_{ij} \boxed{D_{y_j}}}$$

Influence coefficients for the special case of y=0 $$A_{ij} = -\frac{G}{\pi(1-\nu)} \frac{a_j}{(x_i-x_j)^2 - a_j^2}$$

It is noteworthy that the central point used by Crouch in his DD element formulation represents the first Gauss-Chebyshev integration point.

BC’s of the uniformly pressurized crack problem

$$\sigma_{xy} = 0 \qquad -\infty \le x \le \infty$$

$$\sigma_{yy} = -p \qquad |x| < 1,\ y = 0$$

$$u_y = 0 \qquad |x| \ge 1,\ y = 0$$

CDD approximate solution

$$\sum_{j=1}^{N} A_{ij}\, D_{y_j} = -p, \qquad i = 1,\ldots,N$$

Eq. (9) is an approximation of the following Singular Integral Equation of the 1st kind

Hadamard finite-part integral

$$\frac{G}{\pi(1-\nu)} \int_{-1}^{1} \frac{D_y(\tau)\,d\tau}{(x_i-\tau)^2} = -p$$

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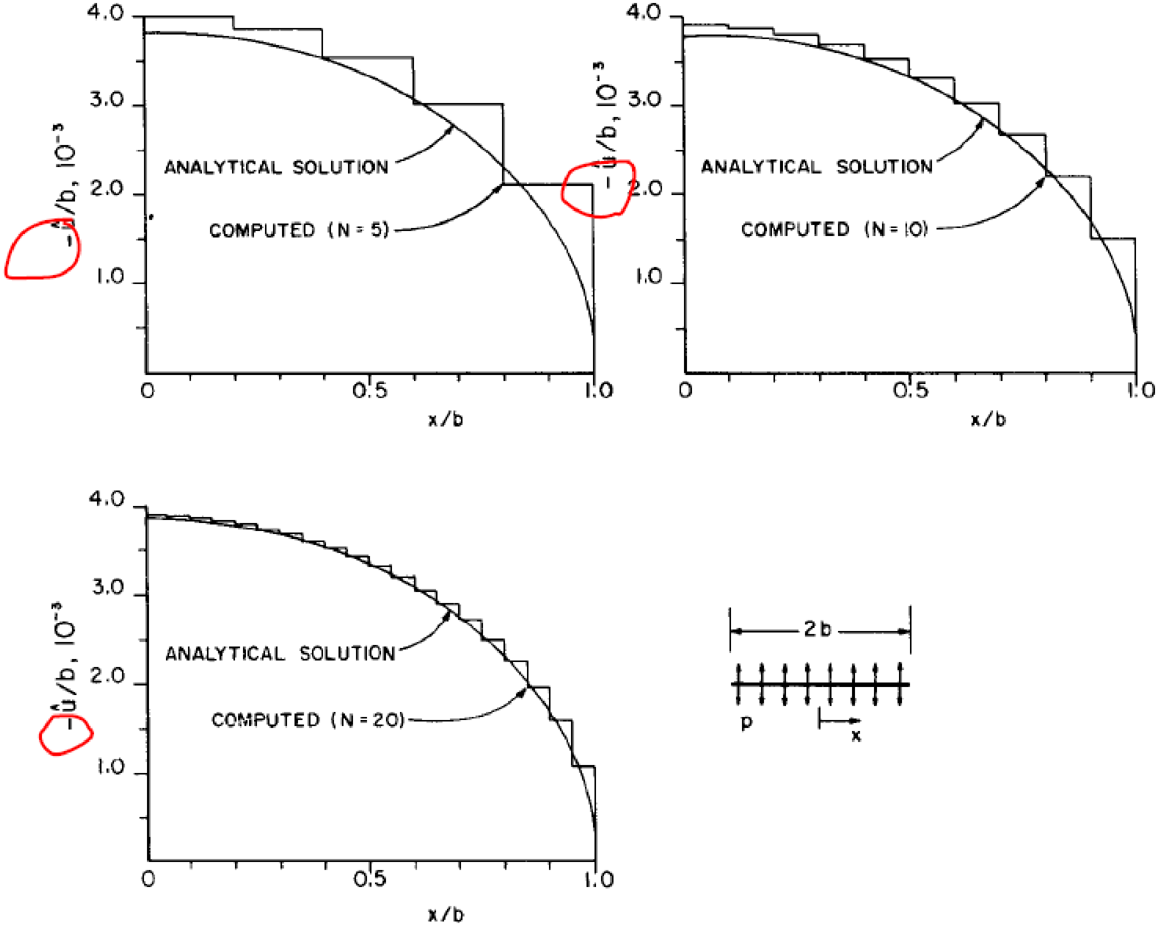


Figure 1. Approximate and exact distributions of relative normal displacement across the surfaces of a pressurized crack

Overestimation of the relative displacements between the crack surfaces in the Crack tip region is a consequence of assuming that the normal stress at the mid-point of the *i-th* line segment represents the average stress over the segment.

Consider the extreme case in which the crack is modeled by one constant DD element of width 2

$$\sigma_{yy}(0,0) = \frac{G}{\pi(1-\nu)} D_y = -p \Rightarrow D_y = -\frac{\pi(1-\nu)p}{G}$$

This solution for the opening of the crack may be compared with the exact solution

$$D_y = -\frac{2(1-\nu)}{G} p$$

The numerical solution overestimates the maximum opening of the crack in this case by $\pi/2$

In turn, overestimation of relative displacements of crack lips means overestimation of SIF’s!!!....see Eqs below

$$K_I = -\frac{G}{4(1-\nu)} \lim_{r\to 0}\left\{\sqrt{\frac{2\pi}{r}} D_y(r)\right\},$$

$$K_{II} = -\frac{G}{4(1-\nu)} \lim_{r\to 0}\left\{\sqrt{\frac{2\pi}{r}} D_x(r)\right\},$$

$$K_{III} = -\frac{G}{4} \lim_{r\to 0}\left\{\sqrt{\frac{2\pi}{r}} D_z(r)\right\}$$

Perspectives

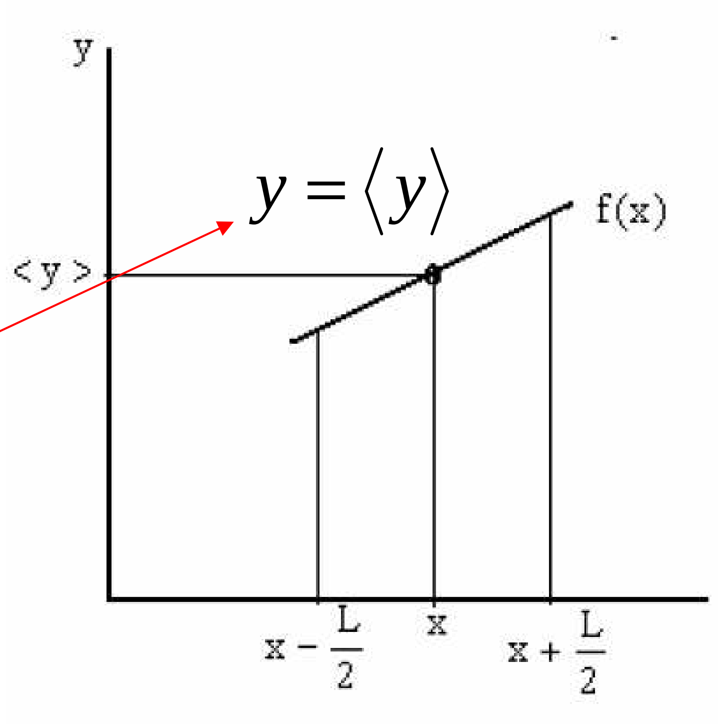
1. Avoid more elaborate elements with more than one collocation points.
2. Simply you need a **better measure of the average stress** at each discontinuity location than the simple midpoint value used.

The impact of non-linearly varying local stress fields on the constitutive law for the stress

$$\langle y\rangle = \frac{1}{L}\int_{-L/2}^{+L/2} f(x+\xi)\,d\xi$$

For linearly varying fields:

$$f(x+\xi)\approx f(x)+f'(x)\xi+o\left(\xi^2\right)$$



For quadratically varying fields:

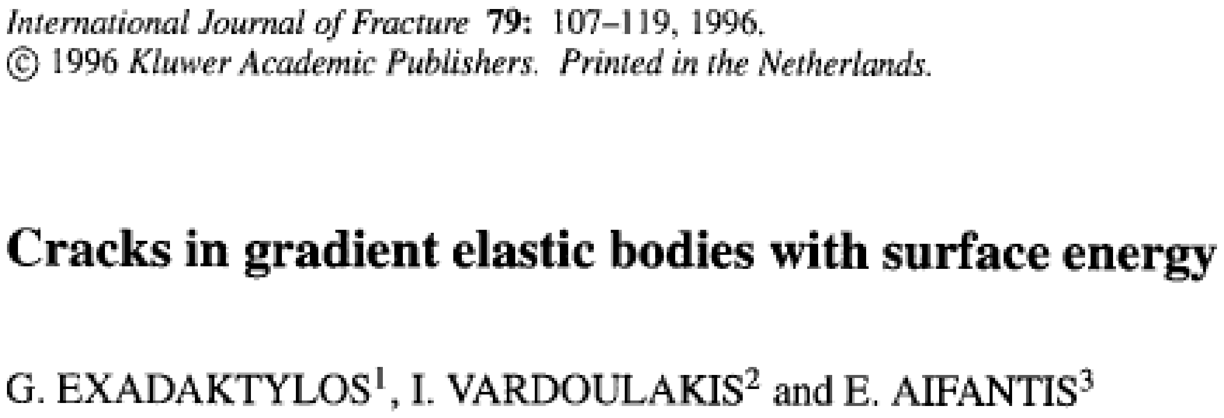
$$f(x+\xi)\approx f(x)+f'(x)\xi+\frac{1}{2}f''(x)\xi^2$$

$$\langle y\rangle = y+\frac{1}{24}L^2\left(\frac{d^2y}{dx^2}\right)_x \Leftrightarrow$$

$$\boxed{y\approx\left(1-\frac{1}{24}L^2\frac{d^2}{dx}\right)\langle y\rangle}$$

Field theories, which are based on averaging rules that include the effect of higher gradients, are called *higher gradient theories*. In particular above rule (B.5) represents a 2nd gradient rule, and can be readily generalized in 2D and 3D by introducing the Laplacian operator instead of the second derivative.

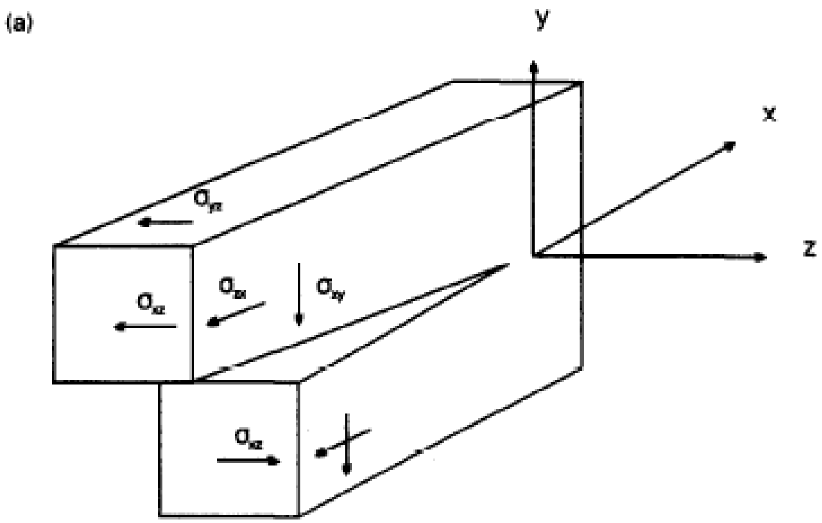
Educational example: Prescribed profile of a Mode III crack



International Journal of Fracture **79:** 107–119, 1996.

Cracks in gradient elastic bodies with surface energy

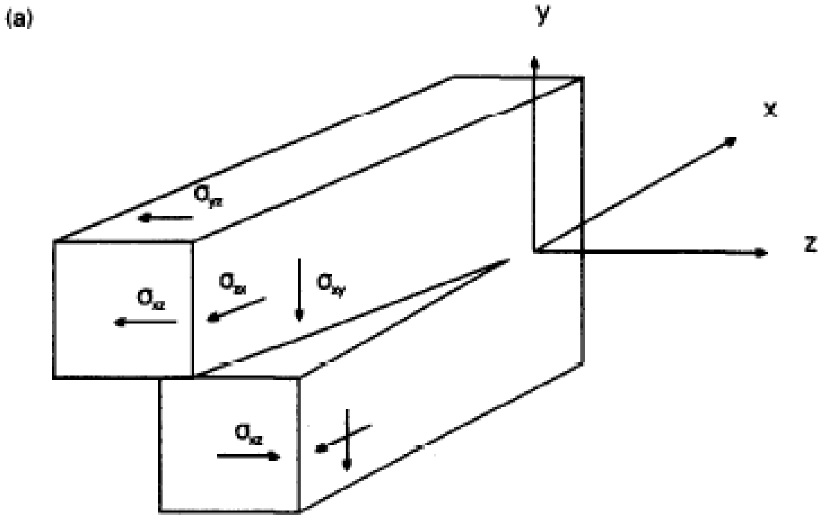
G. EXADAKTYLOS[1], I. VARDOULAKIS[2] and E. AIFANTIS[3]



Simple G2 theory

$$\boxed{\left.\begin{aligned}\sigma_{xz} &= 2G(\varepsilon_{xz} - \ell^2\nabla^2\varepsilon_{xz}),\\ \sigma_{yz} &= 2G(\varepsilon_{yz} - \ell^2\nabla^2\varepsilon_{yz})\end{aligned}\right\}}$$

$$\varepsilon_{xz} = \frac{1}{2}\frac{\partial w}{\partial x}, \quad \varepsilon_{yz} = \frac{1}{2}\frac{\partial w}{\partial y}$$



Prescribed profile of a Mode III crack

$$w(x,0) = D_z\left(1 - x^2\right)^c H(1-x), \; c \geq 0$$

Extra BC

$$\frac{\partial^2 w}{\partial y^2} = 0, \quad -\infty < x < \infty, \quad y = 0$$

Fourier integral transform:

$$F_c[f(x); x \to \xi] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos x\xi d\xi$$

$$\left\{\begin{aligned}
w(x,y) &= F_c\left[A(\xi)e^{-y\xi} + B(\xi)e^{-y\sqrt{\xi^2+1/\ell^2}}; \xi \to x\right],\\
\sigma_{xz}(x,y) &= -GF_s\left[\xi A(\xi)e^{-y\xi}; \xi \to x\right],\\
\sigma_{yz}(x,y) &= -GF_c\left[\xi A(\xi)e^{-y\xi}; \xi \to x\right],\\
\frac{\partial^2 w}{\partial y^2}(x,y) &= F_c\left[\xi^2 A(\xi)e^{-y\xi} + \right.\\
&\qquad \left. + \frac{1}{\ell^2}\left(1 + \ell^2\xi^2\right)B(\xi)e^{-y\sqrt{\xi^2+1/\ell^2}}; \xi \to x\right].
\end{aligned}\right.$$

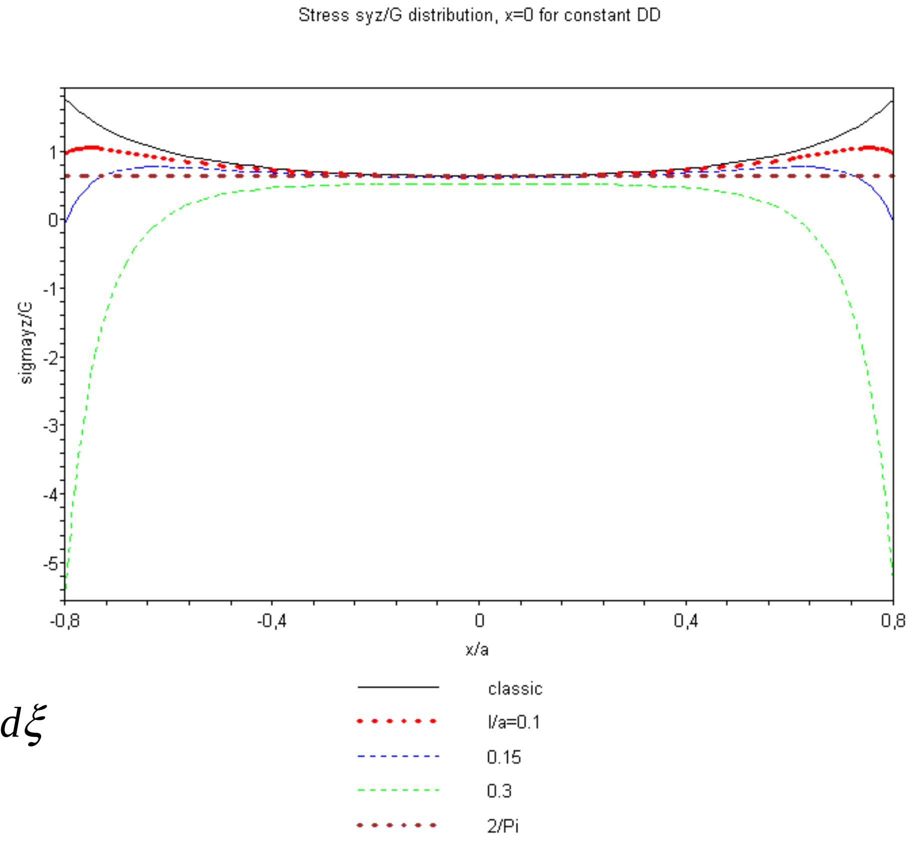
Solution of the single straight Mode III crack problem

BC #1.

$$A(\xi)+B(\xi)=F_c\left[D_z\left(1-x^2\right)^c H(1-x);x\to\xi\right]$$

BC #2. $\dfrac{\partial^2 w}{\partial y^2}=0,\quad -\infty<x<\infty,\quad y=0$

$$\Rightarrow B(\xi)=-\frac{\xi}{a(\xi)}A(\xi),\quad 0\le\xi<\infty$$



$$\frac{\sigma_{yz}(x,0)}{G}=-\sqrt{\frac{2}{\pi}}\,2^c\Gamma(c+1)D_z\int_0^\infty \xi^{1/2-c}\left(1+\ell^2\xi^2\right)J_{c+1/2}(\xi)\cos x\xi\,d\xi$$

$$\frac{\sigma_{yz}(x,0)}{G}=-\sqrt{\frac{2}{\pi}}\,2^c\Gamma(c+1)D_z\left\{\frac{2^{\frac{1}{2}-c}\,{}_2F_1\left(1,\frac{1}{2}-c;\frac{1}{2};x^2\right)}{\Gamma\left(c+\frac{1}{2}\right)}+\ell^2\frac{2^{\frac{5}{2}-c}\,{}_2F_1\left(2,\frac{3}{2}-c;\frac{1}{2};x^2\right)}{\Gamma\left(c-\frac{1}{2}\right)}\right\},\quad c\ge 0.$$

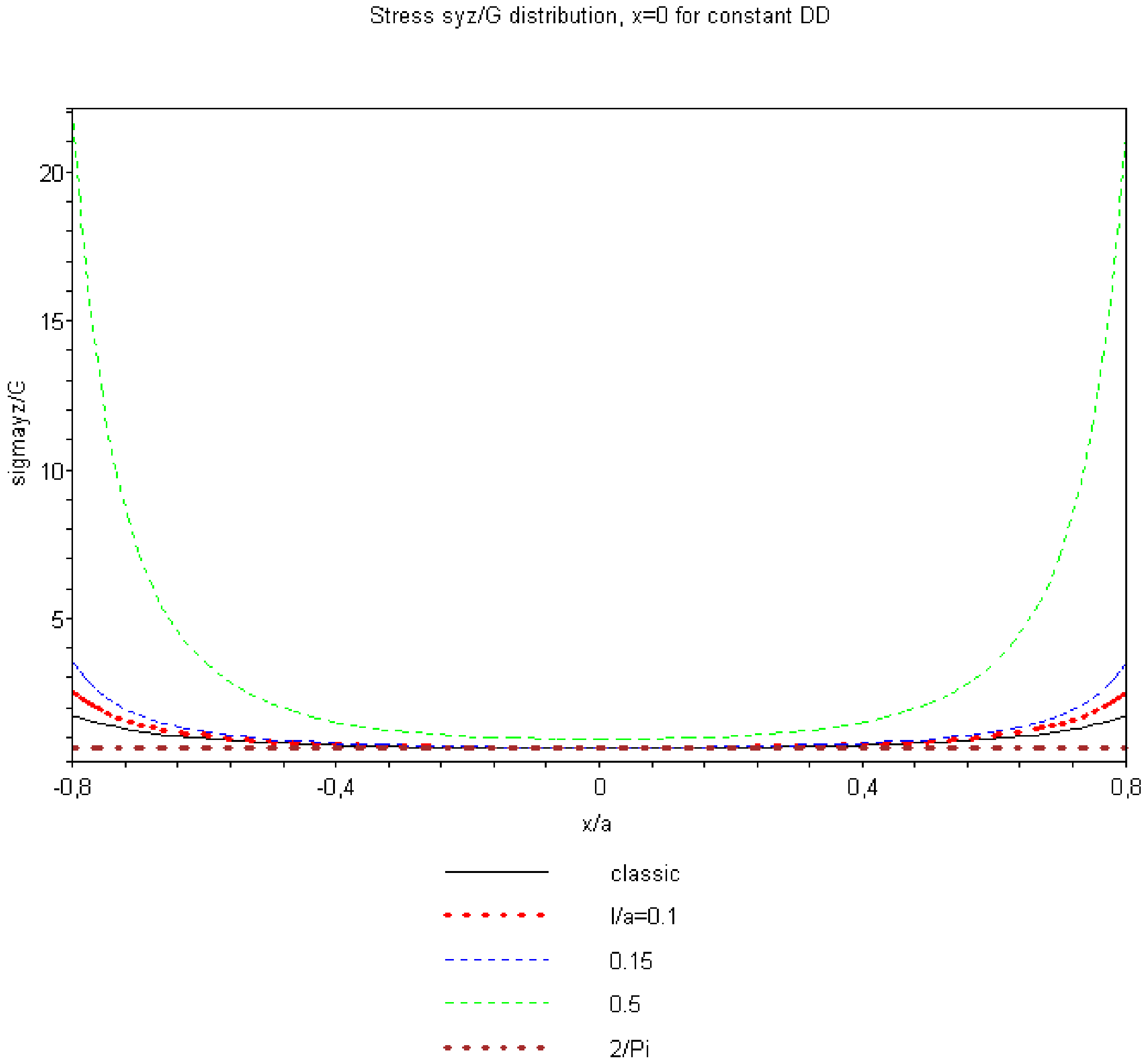
For c=0:

$$w(x,0)=D_z\;H(1-x)=(ct)$$

$$\frac{\sigma_{yz}(x,0)}{G}=\frac{1}{\pi}D_z\left(\frac{1}{1-x^2}-2\ell^2\,{}_2F_1\left(\frac{3}{2},2;\frac{1}{2};x^2\right)\right)$$

$$\ell = i\ell, \quad i \equiv \sqrt{-1}$$

$$\frac{\sigma_{yz}(x,0)}{G} = \frac{1}{\pi} D_z \left(\frac{1}{1-x^2} + 2\ell^2 {}_2F_1\left(\frac{3}{2},2;\frac{1}{2};x^2\right) \right)$$



In the sequel we seek that value of the length scale that gives the exact agreement of the mid-point displacement of the uniformly pressurized CDD with the analytical solution for the uniformly pressurized crack,assuming that the latter is discretized with only one element.

$$\sigma_{yz}(0,0) = -T \Leftrightarrow \frac{D_z G}{\pi}\left[1+2|\ell|^2\right] = -T \Leftrightarrow D_z = -\frac{T}{G}\frac{\pi}{\left[1+2|\ell|^2\right]}$$

The above value of DD should be equal to the mid-point opening of the Mode-I crack as it given from Eq. (2)

$$-\frac{T}{G}\frac{\pi}{\left[1+2|\ell|^2\right]} = -2\frac{T}{G}$$

Solution!..

$$|\ell| = \frac{1}{2}\sqrt{\pi - 2}$$

Construction of the new G2CDD element (G2 stands for grade-2 or 2nd gradient theory)

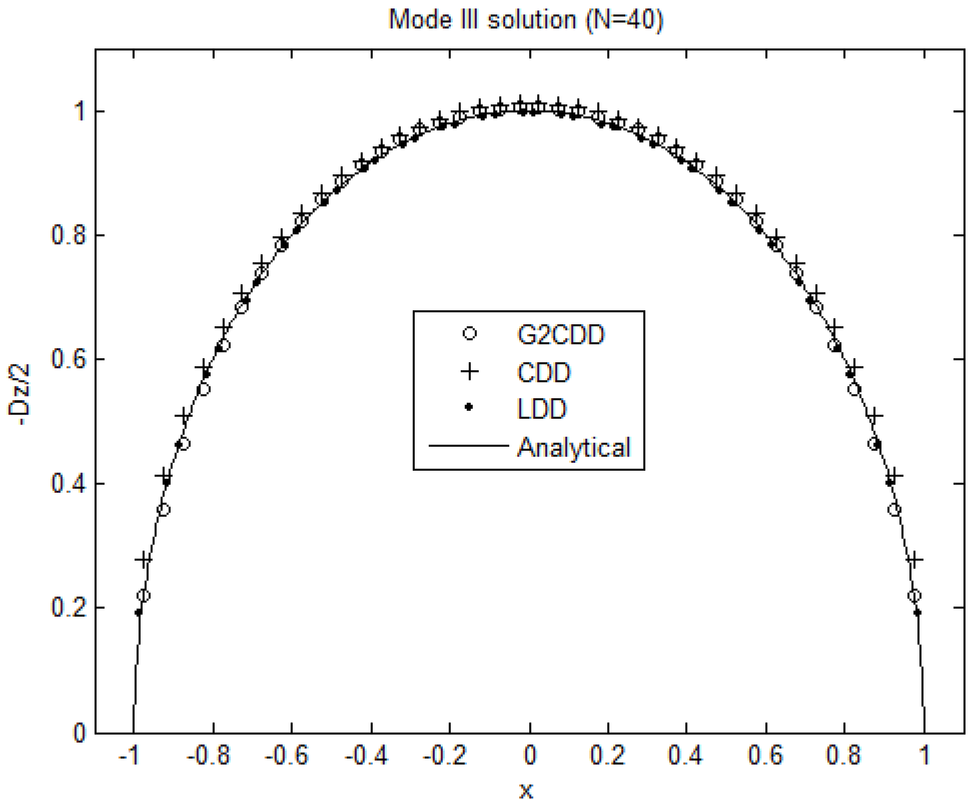
Creation of new Influence Functions !!!....

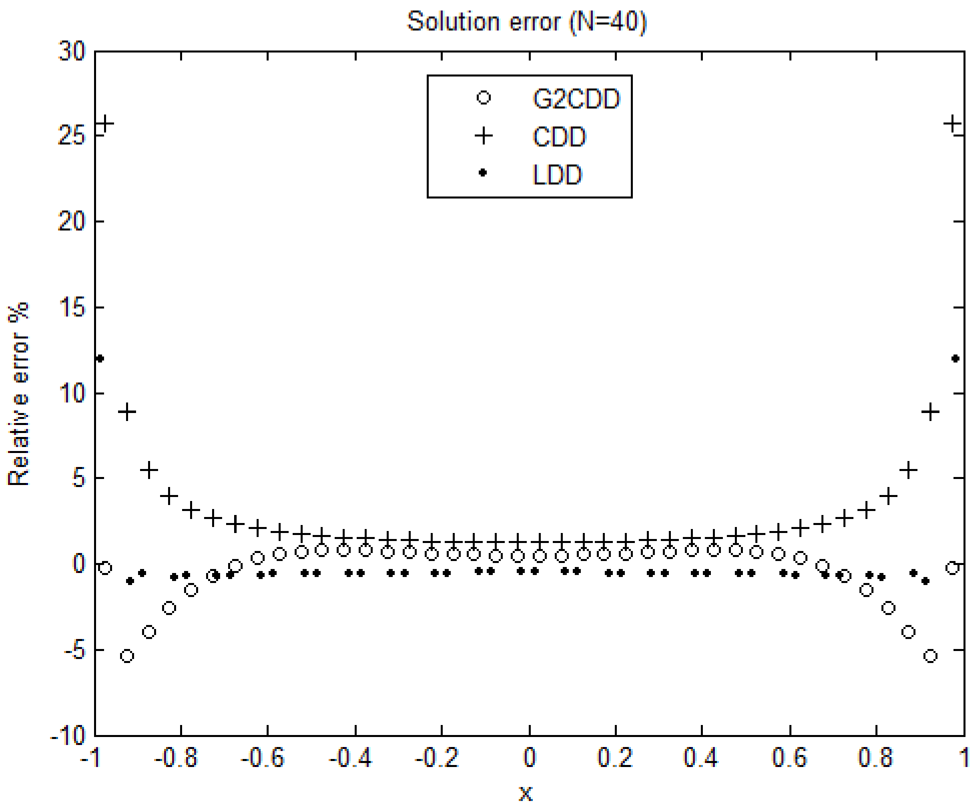
$$\sigma_{yz}\left(x_i,0\right)=\sigma_{yz_i}=A_{ij}D_{z\,j}$$

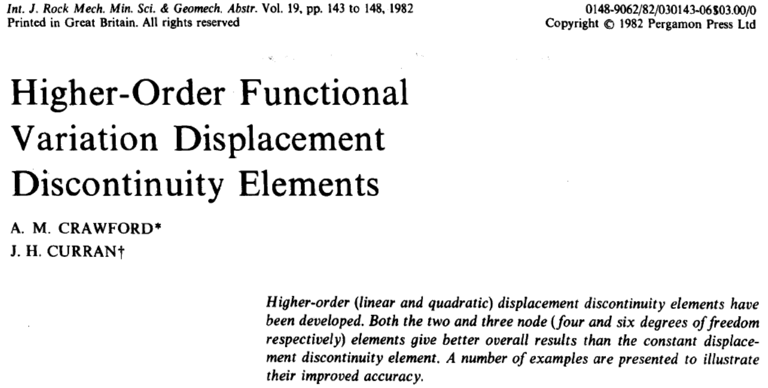
$$A_{ij}=-\frac{a_j G}{\pi}\frac{1}{\left(x_i-x_j\right)^2-a_j^2}\left\{1+\boxed{2\frac{|\ell|^2}{a_j^2}\frac{\left[a_j^2+3\left(x_i-x_j\right)^2\right]^2}{\left[\left(x_i-x_j\right)^2-a_j^2\right]^2}}\right\}$$

New G2-term

Comparison of G2CDD with CDD & LDD: 1st case of Mode III crack problem under uniform shear






Int. J. Rock Mech. Min. Sci. & Geomech. Abstr. Vol. 19, pp. 143 to 148, 1982
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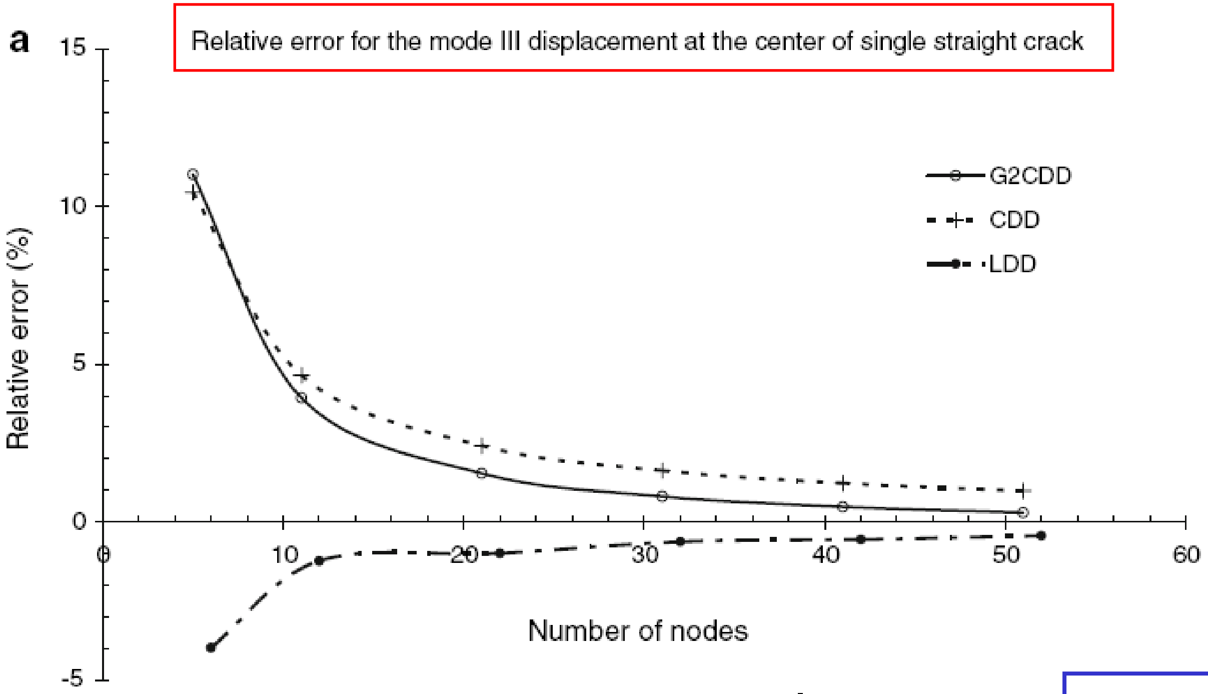
Higher-Order Functional Variation Displacement Discontinuity Elements

A. M. CRAWFORD*
J. H. CURRAN†

Higher-order (linear and quadratic) displacement discontinuity elements have been developed. Both the two and three node (four and six degrees of freedom respectively) elements give better overall results than the constant displacement discontinuity element. A number of examples are presented to illustrate their improved accuracy.

Relative error for the mode III displacement at the center of single straight crack



Relative error for the mode III SIF for the single straight crack

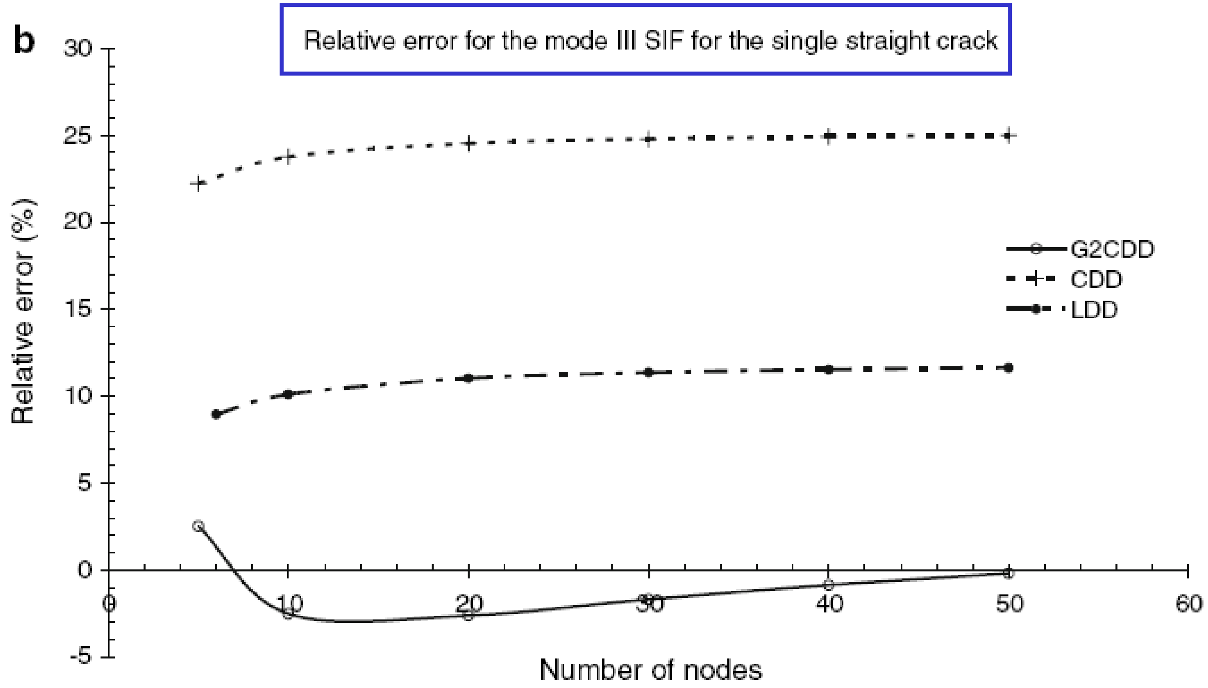
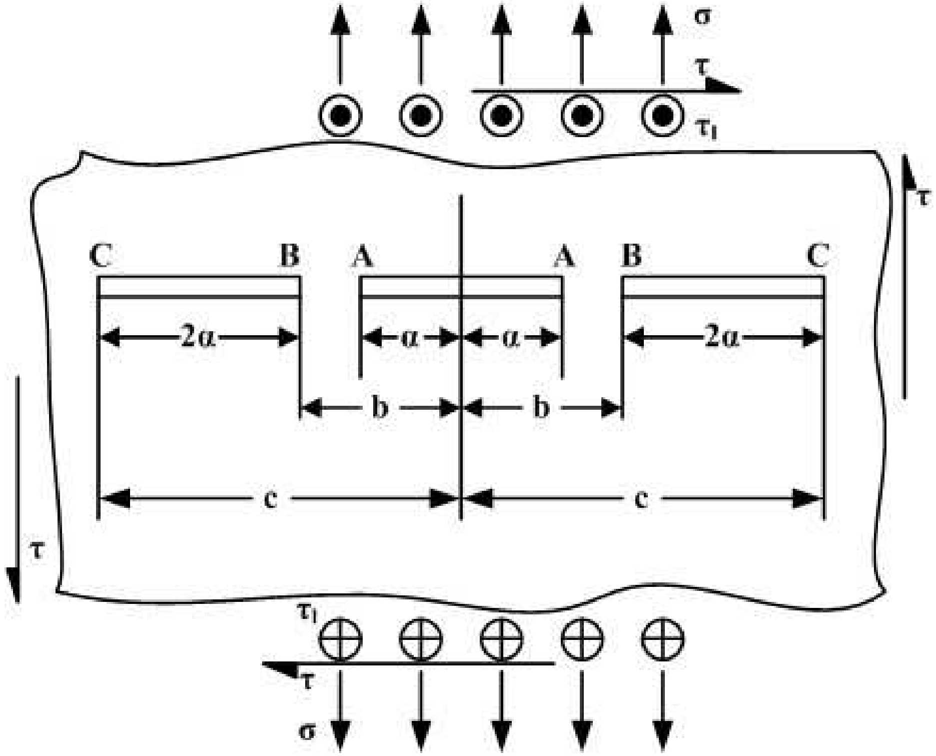


Fig. 6 Variation of the relative error of (**a**) the displacement at the centre of the crack, and (**b**) the Mode III SIF, for the single straight Mode III crack problem

The case of three co-linear straight cracks



Analytical solution:

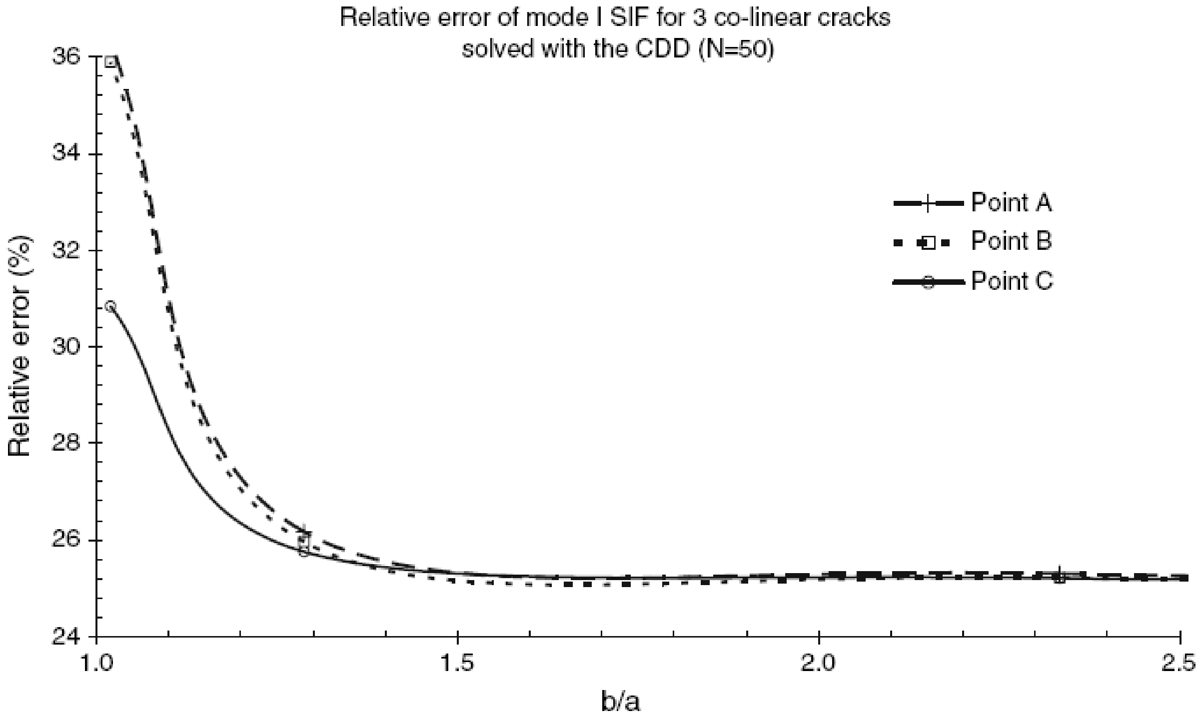
$$\begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix}_A = \begin{Bmatrix} \sigma \\ \tau \\ \tau_l \end{Bmatrix} \sqrt{\pi a}\sqrt{\frac{c^2-a^2}{b^2-a^2}}\frac{E(k)}{K(k)},$$

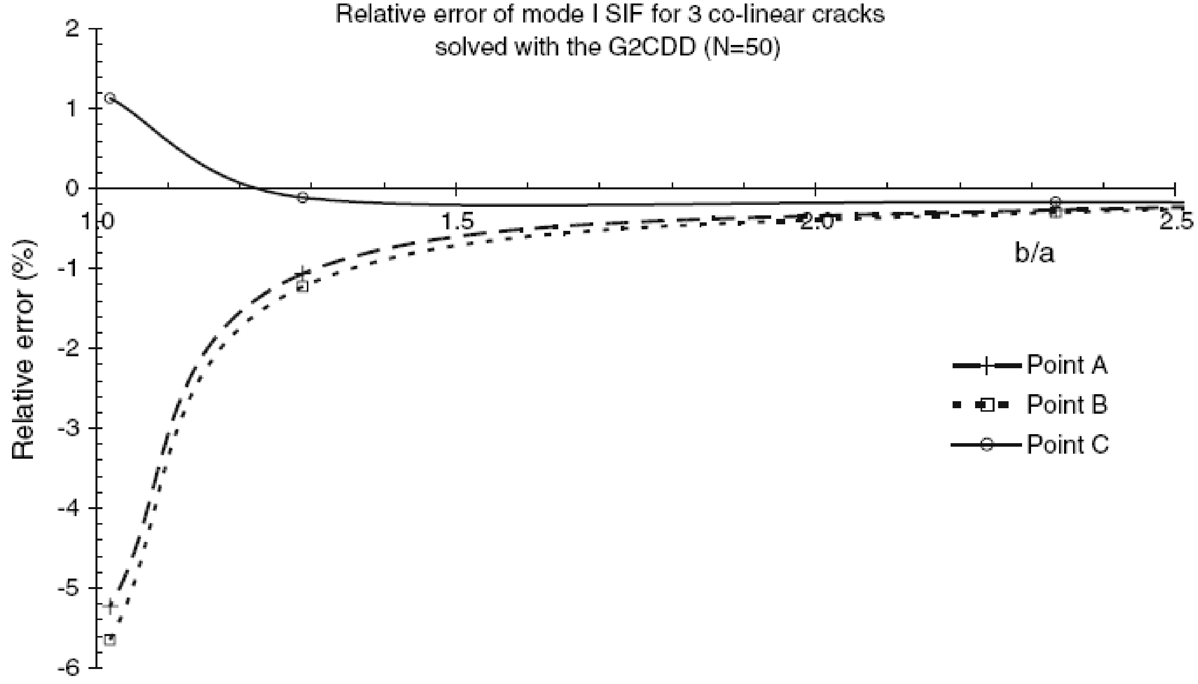
$$\begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix}_B = \begin{Bmatrix} \sigma \\ \tau \\ \tau_l \end{Bmatrix} \sqrt{\pi b}\sqrt{\frac{b^2-a^2}{c^2-b^2}}\left(\frac{c^2-a^2}{b^2-a^2}\frac{E(k)}{K(k)}-1\right),$$

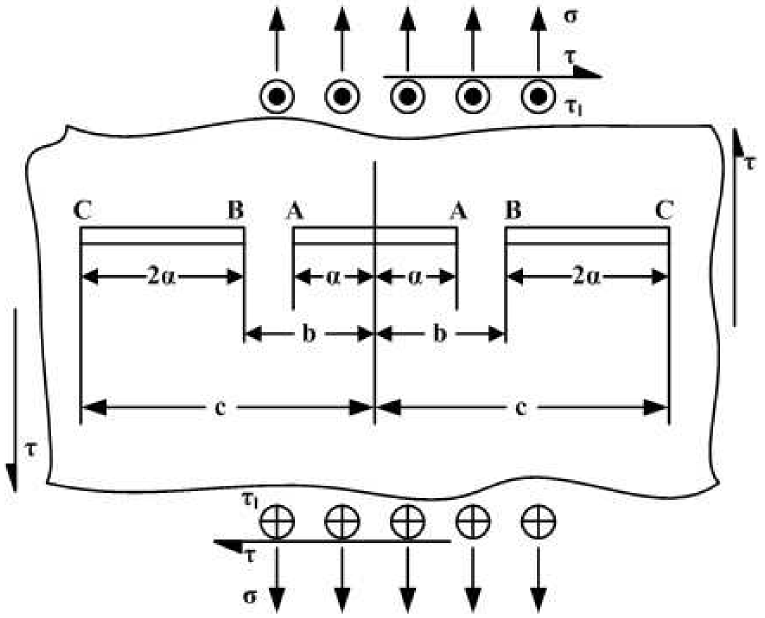
$$\begin{Bmatrix} K_I \\ K_{II} \\ K_{III} \end{Bmatrix}_C = \begin{Bmatrix} \sigma \\ \tau \\ \tau_l \end{Bmatrix} \sqrt{\pi c}\sqrt{\frac{c^2-a^2}{c^2-b^2}}\left(1-\frac{E(k)}{K(k)}\right)$$

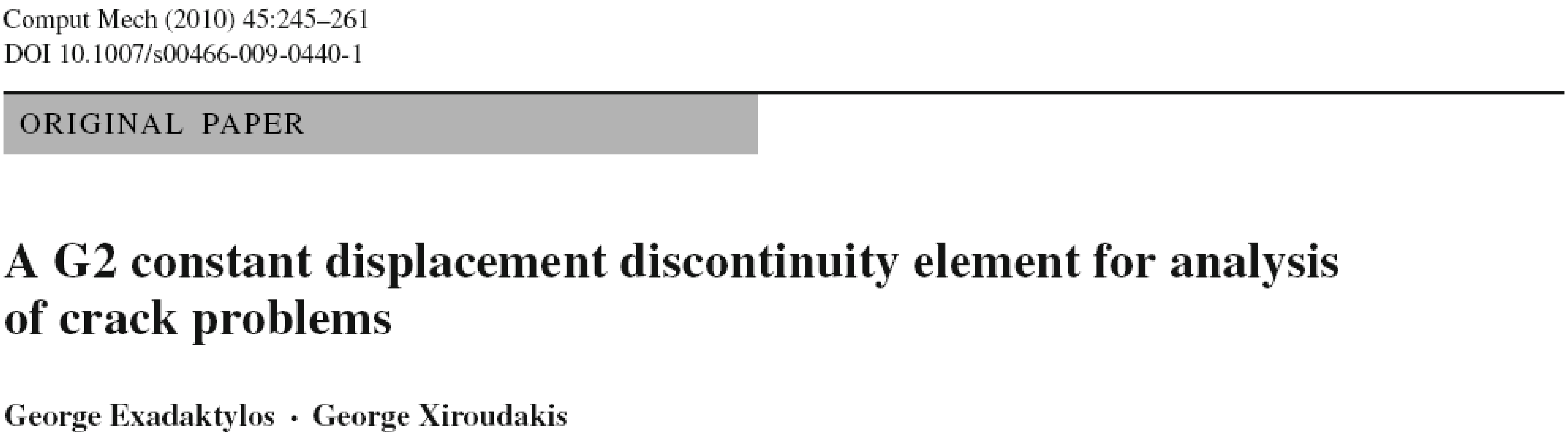
$$k^2 = \left(c^2-b^2\right)\big/\left(c^2-a^2\right)$$

Fig. 12 Dependence of the relative error of the Mode I SIF at the three crack tips A, B and C computed with the CDDM on the ratio b/a


Relative error of mode I SIF for 3 co-linear cracks solved with the CDD (N=50)


Relative error of mode I SIF for 3 co-linear cracks solved with the G2CDD (N=50)





Comput Mech (2010) 45:245–261
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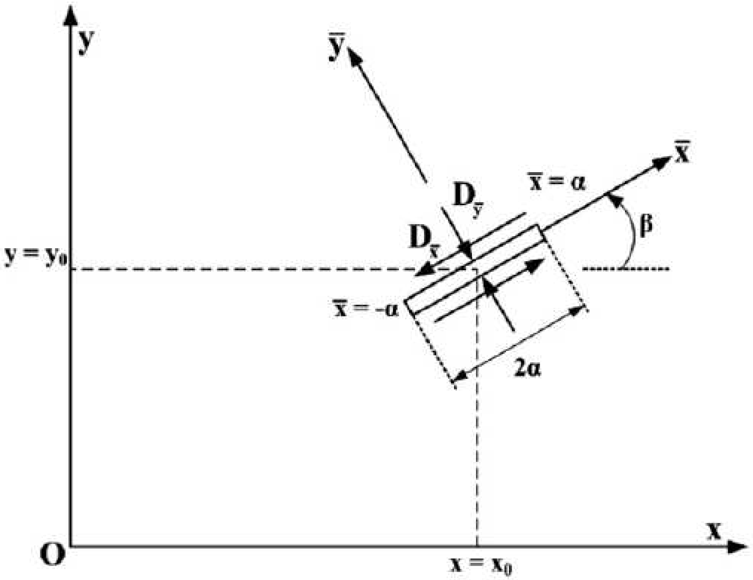
ORIGINAL PAPER

A G2 constant displacement discontinuity element for analysis of crack problems

George Exadaktylos · George Xiroudakis

Generalization for multiple cracks of any shape



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SOLIDS AND STRUCTURES

The G2 constant displacement discontinuity method – Part I: Solution of plane crack problems

G. Exadaktylos *, G. Xiroudakis

Mining Engineering Design Laboratory, Department of Mineral Resources Engineering, Technical University of Crete, GR-73100 Chania, Greece

Validation of G2CDD against known results

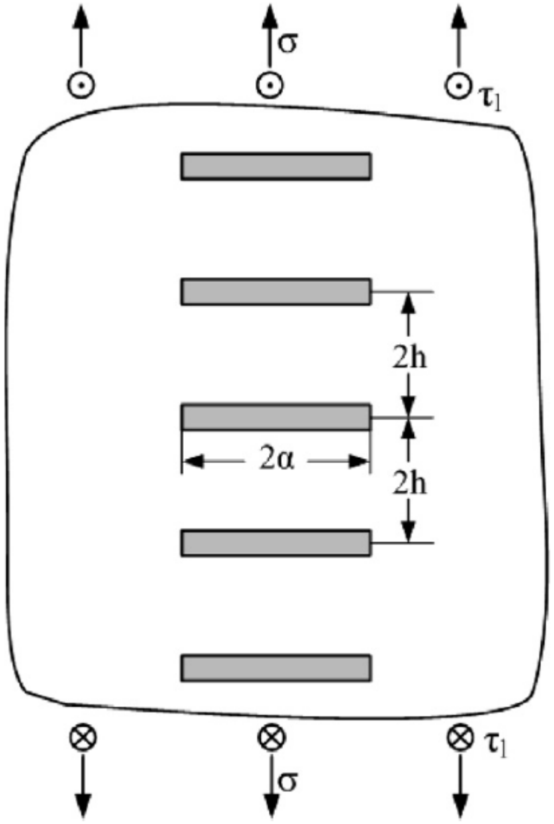

Fig. 4. Periodic array of parallel straight cracks in an infinite solid subjected to far-field uniaxial tensile stress and anti-plane shear stress.

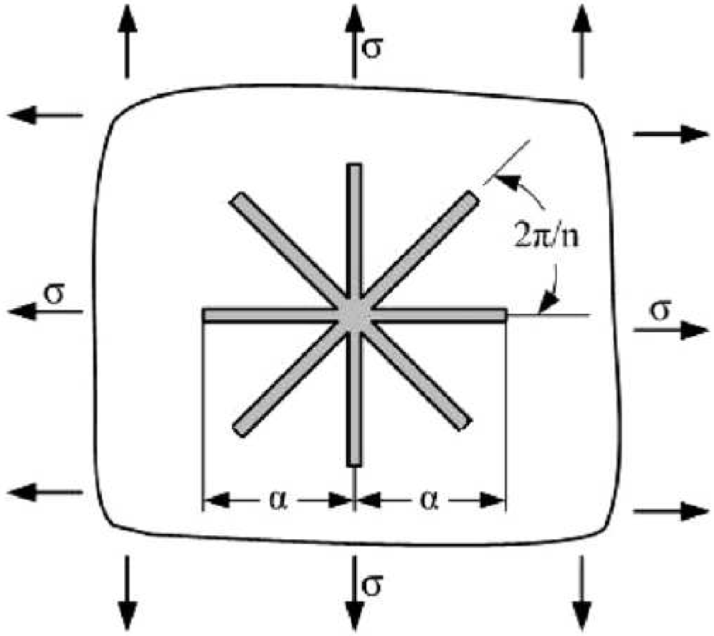

Fig. 6. Radial straight cracks of equal length emanating from a common point in an

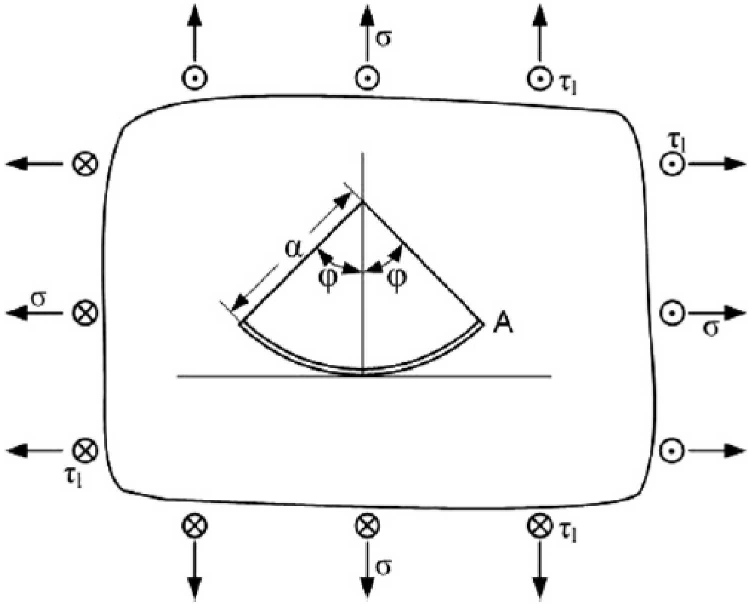

Fig. 8. The curved crack which is a part of a circle in an infinite isotropic plane subjected to biaxial tensile and anti-plane shear stresses.

Interaction of multiple cracks with a free surface

This solution is constructed by superposition from the infinite body results presented in Part I by using the classical method of images (Hirth and Lothe, 1982).

G. Exadaktylos, G. Xiroudakis / International Journal of Solids and Structures 47 (2010) 2578–2590

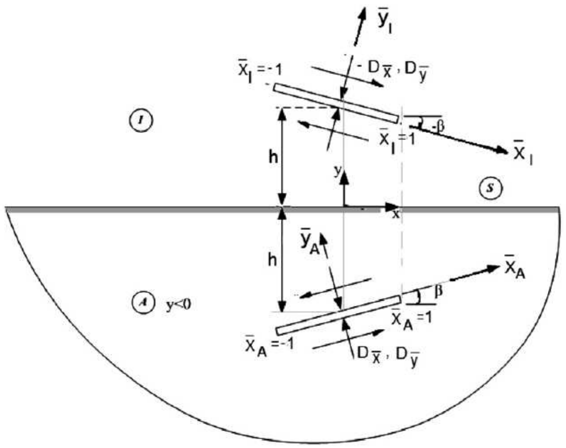

Fig. 1. Arbitrarily oriented finite line segment in the lower half-space with its image in the upper half-plane and coordinate systems.

$$\sigma_{ij} = \sigma_{ij(A)} + \sigma_{ij(I)} + \sigma_{ij(S)} \qquad i.j = x, y$$

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The G2 constant displacement discontinuity method – Part II: Solution of half-plane crack problems

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Mining Engineering Design Laboratory, Department of Mineral Resources Engineering, Technical University of Crete, GR 73100, Chania, Greece

Validation of G2CDD against known results

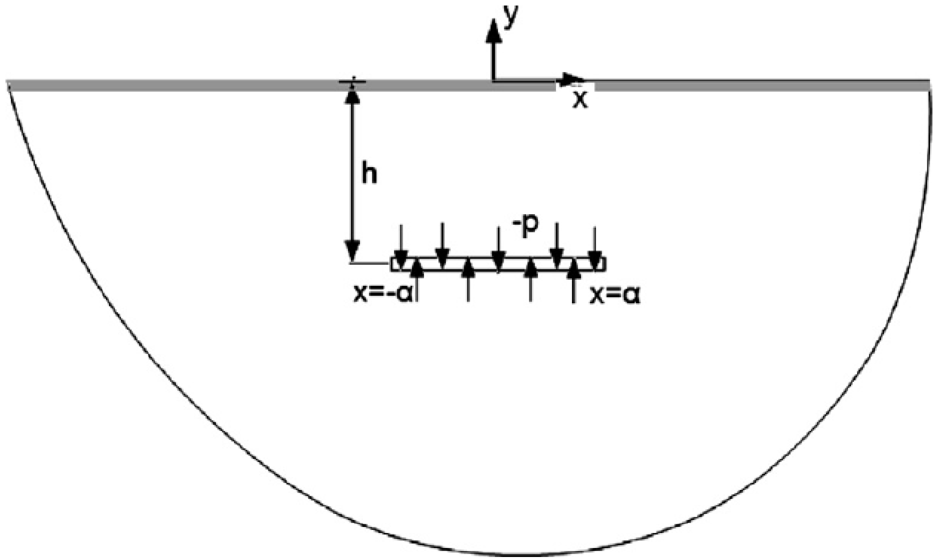

Fig. 2. Geometry and coordinate system for the uniformly pressurized horizontal crack parallel to the free surface and lying in the lower half-plane.

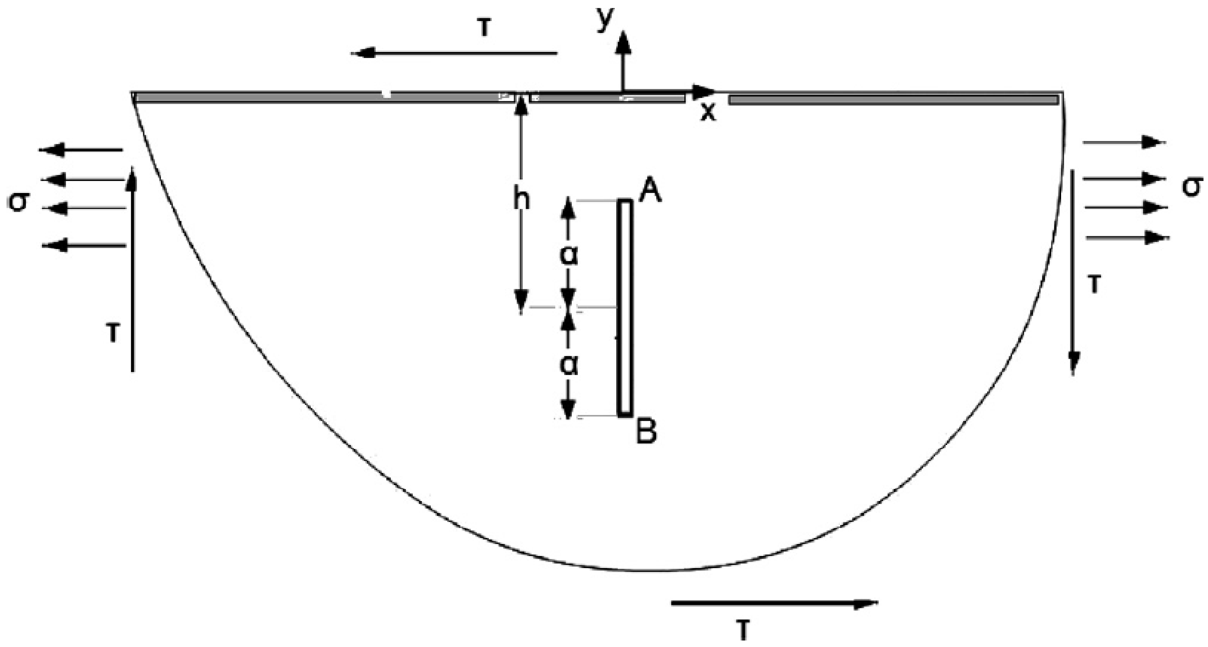

Fig. 5. Straight crack of length 2α normal to the free surface of the half-plane subjected to far-field uniform horizontal tension σ and in-plane shear stress τ.

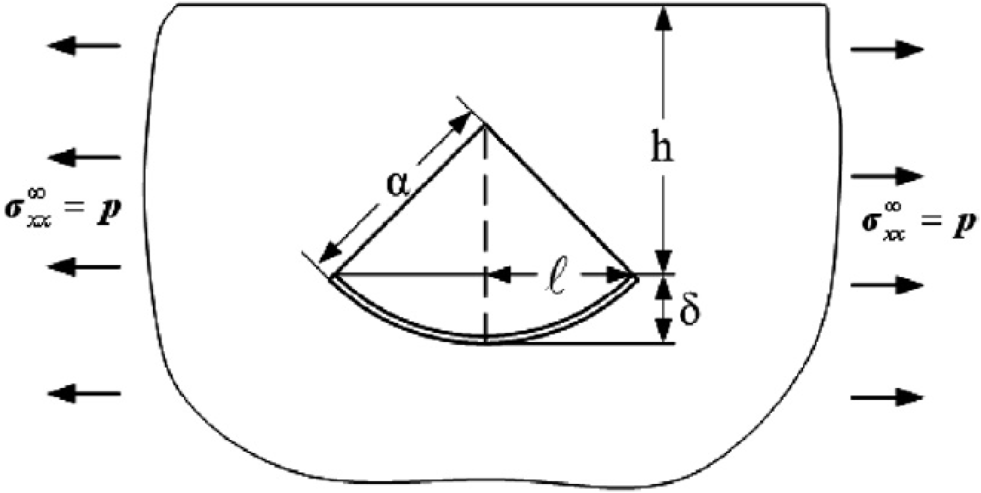

Fig. 8. A curvilinear crack along a part of a circle in the half-plane subjected to far-field horizontal tension.

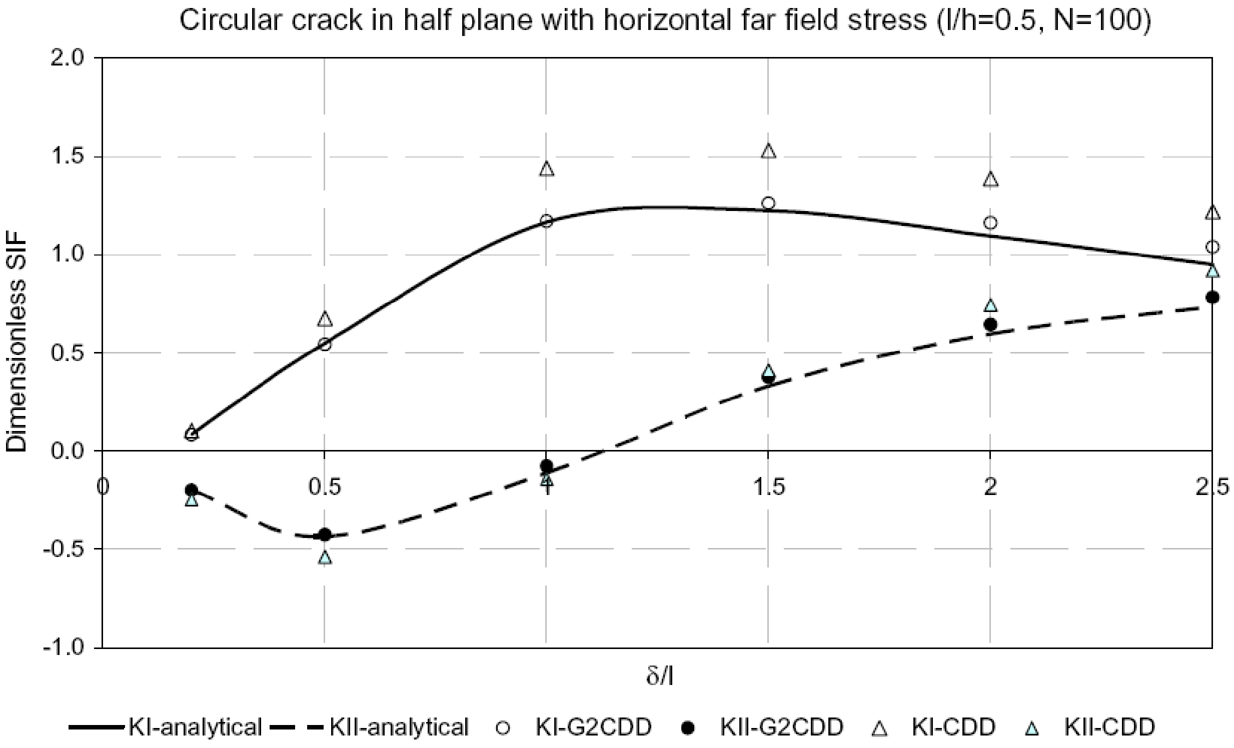

Fig. 9. Variation of modes I and II SIFs with the dimensionless parameter $\delta/\ell$for fixed number of discretization elements (N = 100) and relative depth of the crack ℓ/h = 0.5 as is predicted by the singular integral formulation (referred as "analytical"here) as well as the G2CDD and CDD methods.

Conclusions

- A brief account of existing analytical methods for attacking LEFM problems has been made.
- A new CDD element was presented for the numerical solution of Mode I, II and III crack problems, based on the strain gradient elasticity theory in its simplest possible Grade-2 (second gradient of strain or G2 theory) variant.
- It has the advantage of simplicity, yet it has rather good accuracy appropriate for the fast solution of multiple crack problems.