

Continua with Microstructure Part II: Second-gradient Theory

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Outline

Introduction

Gradient Elasticity

Other gradient theories

Example problems

Numerical implementation

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Introduction

Continua with microstructure

Introducing second-gradient models

Gradient Elasticity

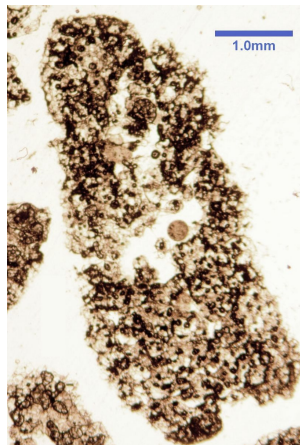
Other gradient theories

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Numerical implementation

Microstructure of materials

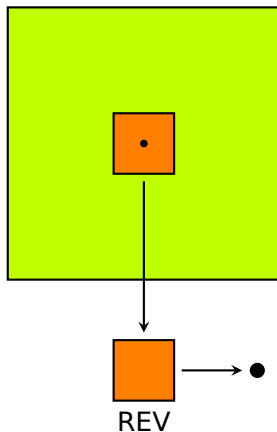
- ▶ Materials always have a microstructure. . .
- ▶ . . . possibly at different length scales
- ▶ Overall **mechanical** behaviour depends on mechanical behaviour of microstructure
- ▶ Usually “whole” is less than sum of “parts” (continuum **averaging**)



Grain of expanded perlite
(Georgopoulos, 2006)

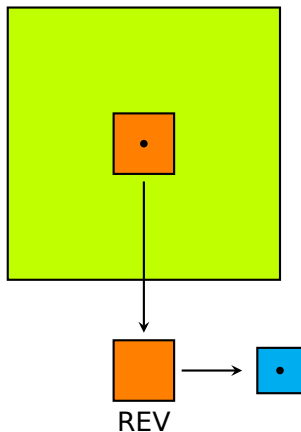
Classical Continuum

- ▶ Discrete microstructure to continuum model
- ▶ Representative elementary volume (REV) associated to material point
- ▶ REV has finite size
- ▶ Material “point” in classical continuum is **dimensionless** (no microstructure)



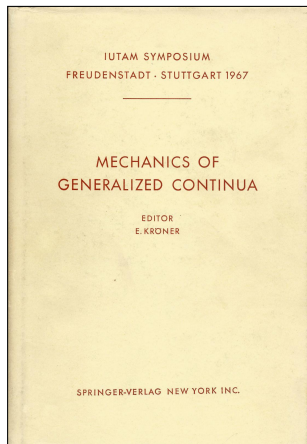
Continuum with microstructure

- ▶ Material “point” has microstructure, finite dimensions
- ▶ Introduction of **internal length** as material parameter
- ▶ Comparison between problem size and material size
- ▶ **Interaction** between different length scales



Generalised continuum theories

- ▶ Different theories: Cosserat, second-gradient, non-local, micromorphic, ...
- ▶ Mechanics of generalised continua is "... the mechanics of matter for circumstances in which the conventional continuum mechanics cannot offer a **promising** approach" (Kröner, 1968)
- ▶ Clear **qualitative** difference from "classical" continuum



Comparison with classical continuum

What is being generalised?

A generalised continuum may:

1. Violate **Cauchy's postulate** (traction on a face is linear in the normal to the face)
2. Refer to non-Euclidean or not-connected space or body
3. Have applied point couples in volume or surface
4. Have additional internal degrees of freedom describing "**microstructure**"

See G. Maugin, "Generalized Continuum Mechanics: What Do We Mean by That?" (2010)

Second-gradient theory

The basic concept

- ▶ Additional dependence of mechanical behaviour on the **second** spatial derivative of displacement
- ▶ Can be seen as dependence on the strain and the (first) strain gradient
- ▶ Also called “first strain gradient”, “strain gradient”, “second grade”, “grade two” or “gradient” theory
- ▶ Used in problems of elastic deformation, fracture, **strain localisation**

Cauchy's higher-gradient concept

«Note sur l'équilibre et les mouvements vibratoires des corps solides», 1851:

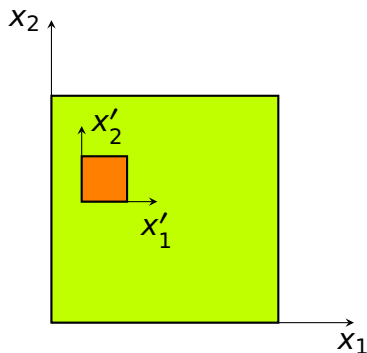
*... les composants $A, F, \mathcal{E}; \mathcal{F}, \mathcal{B}, \mathcal{D}; \mathcal{E}, \mathcal{D}, \mathcal{C}$ des pressions supportées au point P par trois plans parallèles aux plans coordonnés des yz , des zx et des xy , pourront être généralement considérées comme des fonctions linéaires des déplacements ξ, η, ζ et des leurs **dérivées des divers ordres**.*

The “stresses” are considered as linear functions of the displacements and their **derivatives of various orders**.

Mindlin's elasticity with microstructure

"Micro-structure in linear elasticity" (1964)

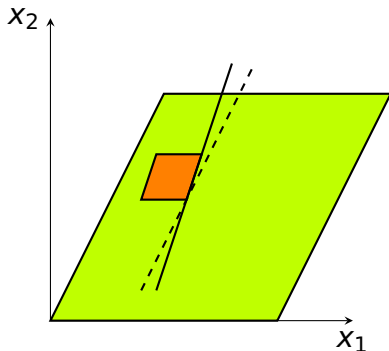
- ▶ A more general theory
- ▶ Each material point has a deformable micro-volume
- ▶ Two scales: **micro**-medium & **macro**-medium
- ▶ Each scale has it's own deformation
- ▶ Second-gradient theory seen as **limit** case for zero relative deformation



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A simple 1D second-gradient model

- ▶ Average $\langle y \rangle$ of field $y = f(x)$ over length L (REV)

$$\langle y \rangle = \frac{1}{L} \int_{-L/2}^{L/2} f(x + \xi) d\xi$$

- ▶ Express $f(x + \xi)$ as Taylor series around $\xi = 0$

$$f(x + \xi) = f(x) + f'(x)\xi + \frac{1}{2}f''(x)\xi^2 + \frac{1}{6}f'''(x)\xi^3 + O(\xi^4)$$

- ▶ Case **1**: local curvature is negligible

$$y \approx \langle y \rangle$$

- ▶ **locally homogeneous** or **local** continuum

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- ▶ Case **2**: local curvature is **not** negligible

$$y \approx \langle y \rangle - \frac{L^2}{24} \frac{d^2 \langle y \rangle}{dx^2}$$

- ▶ **locally inhomogeneous** or **non-local** continuum

A simple 1D second-gradient model

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Second-gradient theory with an internal length

Outline

Introduction

Gradient Elasticity

Introductory concepts

Variational formulation

Linear and isotropic gradient elasticity

Other gradient theories

Example problems

Numerical implementation

Gradient elasticity

- ▶ Work by Toupin (1962), Mindlin (1964), Mindlin & Eshel (1968)
- ▶ Simplest non-trivial second gradient model
- ▶ Shows basic properties of second-gradient theories
- ▶ Allows analytical treatment of simple problems
- ▶ Rigorous mathematical framework

- ▶ Here we assume:
 - ▶ (Quasi-) Static case
 - ▶ Small deformations
 - ▶ Cartesian coordinates

Indexical notation for tensors

- ▶ Tensors are indicated using indices i, j, k, \dots

$$x_i, u_i, \tau_{ij}, K_{ijk}, \dots$$

- ▶ Indices from 1 to 3

$$x_i = \{x_1, x_2, x_3\}$$

- ▶ Repeated indices indicate summation

$$\tau_{ii} = \tau_{11} + \tau_{22} + \tau_{33}$$

- ▶ A comma indicates spatial differentiation

$$u_{i,j} = \partial u_i / \partial x_j$$

- ▶ Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- ▶ Permutation symbol

$$e_{ijk} = \begin{cases} 1 & 123, 231, 312 \\ -1 & 321, 213, 132 \\ 0 & \text{otherwise} \end{cases}$$

Grouping the displacement derivatives

Classical theory

- ▶ Classical theory is first-gradient
- ▶ Mechanical behaviour depends only on $u_{i,j}$
- ▶ Actually only on the symmetric part (**strain**)

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

- ▶ 6 independent components out of 9

- ▶ We can decompose into spherical and deviatoric components

$$\epsilon_{ij} = \frac{1}{3}\epsilon_{kk}\delta_{ij} + \epsilon'_{ij}$$

- ▶ **Convenient** in some applications
- ▶ **Equivalent** to using just the strain

Grouping the displacement derivatives

Second-gradient theory

- ▶ We define:

$$\tilde{K}_{ijk} = u_{k,ij}$$

$$\hat{K}_{ijk} = \epsilon_{jk,i}$$

$$\bar{K}_{ij} = \frac{1}{2} e_{jlm} u_{m,li}$$

$$\bar{\bar{K}}_{ijk} = \frac{1}{3} (u_{k,ij} + u_{j,ki} + u_{i,jk})$$

- ▶ Dependence on $u_{k,ij}$
- ▶ $u_{k,ij}$ symmetric in i, j
- ▶ 18 independent components out of 27

- ▶ Different groupings (decompositions)

$$u_{k,ij} = \tilde{K}_{ijk} \quad (I)$$

$$= \hat{K}_{ijk} + \hat{K}_{jki} - \hat{K}_{kij} \quad (II)$$

$$= \bar{\bar{K}}_{ijk} - \frac{2}{3} e_{ilk} \bar{K}_{jl} - \frac{2}{3} e_{jlk} \bar{K}_{il} \quad (III)$$

- ▶ Three different “Forms” (other possible)
- ▶ All equivalent
- ▶ We'll use Form II

Variational formulation

- ▶ Potential energy density

$$W = W(\epsilon_{ij}, \kappa_{ijk})$$

where $\kappa_{ijk} = \epsilon_{jk,i}$

- ▶ Variation of W

$$\begin{aligned} \delta W &= \frac{\partial W}{\partial \epsilon_{ij}} \delta \epsilon_{ij} + \frac{\partial W}{\partial \kappa_{ijk}} \delta \kappa_{ijk} \\ &= \tau_{ij} \delta \epsilon_{ij} + \mu_{ijk} \delta \kappa_{ijk} \end{aligned}$$

introduces stresses and
double-stresses

- ▶ Total potential energy

$$\mathcal{W} = \int_V W dV$$

- ▶ Equilibrium requires

$$\delta \mathcal{W} = \delta \mathcal{P}$$

where \mathcal{P} the work of
external actions and

$$\delta \mathcal{W} = \int_V (\tau_{ij} \delta \epsilon_{ij} + \mu_{ijk} \delta \kappa_{ijk}) dV$$

Variation of potential energy

- ▶ Start with (V is volume)

$$\delta\mathcal{W} = \int_V (\tau_{ij}\delta\epsilon_{ij} + \mu_{ijk}\delta\kappa_{ijk}) dV$$

- ▶ Calculate (S is surface)

$$\begin{aligned} \delta\mathcal{W} = & \int_S n_j (\tau_{jk} - \mu_{ijk,i}) \delta u_k dS \\ & - \int_V (\tau_{jk} - \mu_{ijk,i})_{,j} \delta u_k dV \\ & + \int_S n_i \mu_{ijk} \delta u_{k,j} dS \end{aligned}$$

- ▶ On S , $\delta u_{k,j}$ is **not** independent of δu_k
- ▶ Introduce **normal** derivative operator

$$D \equiv n_j \frac{\partial}{\partial x_j}$$

and **surface**-gradient operator

$$D_j \equiv \frac{\partial}{\partial x_j} - n_j D$$

Introducing external actions

After calculations (C are edges, $[[\cdot]]$ is jump)

$$\begin{aligned} \delta \mathcal{W} = & - \int_V (\tau_{jk} - \mu_{ijk,i})_j \delta u_k \, dV \\ & + \int_S (n_j (\tau_{jk} - \mu_{ijk,i}) - D_j (n_i \mu_{ijk}) + (D_l n_l) n_j n_i \mu_{ijk}) \delta u_k \, dS \\ & + \int_S n_i n_j \mu_{ijk} D \delta u_k \, dS + \oint_C s_m e_{mlj} [[n_l n_i \mu_{ijk}]] \delta u_k \, dC \end{aligned}$$

This suggests (F_k body forces, P_k surface tractions, R_k surface double-tractions, E_k edge tractions)

$$\delta \mathcal{P} = \int_V F_k \delta u_k \, dV + \int_S P_k \delta u_k \, dS + \int_S R_k D \delta u_k \, dS + \oint_C E_k \delta u_k \, dC$$

Equilibrium equations

- ▶ Since $\delta\mathcal{W} = \delta\mathcal{P}$ we obtain [strong form]

$$(\tau_{jk} - \mu_{ijk,i})_{,j} + F_k = 0 \quad \text{in } V$$

$$P_k = n_j(\tau_{jk} - \mu_{ijk,i}) - D_j(n_i\mu_{ijk}) + (D_l n_l)n_j n_i \mu_{ijk} \quad \text{on } S$$

$$R_k = n_i n_j \mu_{ijk} \quad \text{on } S$$

$$E_k = s_m e_{mlj} [n_l n_i \mu_{ijk}] \quad \text{on } C$$

- ▶ but also [weak form / virtual work equation]

$$\int_V (\tau_{ij} \delta \epsilon_{ij} + \mu_{ijk} \delta \kappa_{ijk}) dV = \int_V F_k \delta u_k dV + \int_S P_k \delta u_k dS + \int_S R_k D \delta u_k dS + \oint_C E_k \delta u_k dC$$

Remarks on equilibrium equations

- ▶ Additional boundary tractions w.r.t. classical continuum (R_k, E_k)
- ▶ P_k **not** according to Cauchy's postulate
- ▶ Weak form "simpler" than strong form*
- ▶ You **cannot** ignore R_k and E_k in either case

- ▶ Setting $\sigma_{jk} = \tau_{jk} - \mu_{ijk,i}$ yields

$$\sigma_{jk,j} + F_k = 0$$

- ▶ Some authors use σ_{jk} as a "Cauchy" or "equilibrium" stress
- ▶ However,

$$P_k \neq n_j \sigma_{jk}$$

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- ▶ However, **Cannot ignore R_k, E_k**

One stress, energy conj. to strain

Linear gradient elasticity

- ▶ Up to now, only $W = W(\epsilon_{ij}, K_{ijk})$ was assumed
- ▶ A quadratic form of W

$$W = \frac{1}{2} C_{ijpq} \epsilon_{ij} \epsilon_{pq} + f_{ijkpq} K_{ijk} \epsilon_{pq} + \frac{1}{2} a_{ijkpqr} K_{ijk} K_{pqr}$$

results in **linear** gradient elasticity

$$\tau_{ij} = C_{ijpq} \epsilon_{pq} + f_{pqrij} K_{pqr}$$

$$\mu_{ijk} = f_{ijkpq} \epsilon_{pq} + a_{ijkpqr} K_{pqr}$$

- ▶ C_{ijpq} , f_{ijkpq} and a_{ijkpqr} are **material parameter** tensors with 21, 108 and 171 independent parameters (due to various symmetries)
- ▶ Total of **300** independent parameters for general anisotropic linear case

Isotropic linear gradient elasticity

- ▶ For isotropic material parameter tensors (invariant w.r.t. rotation)

$$W = \frac{1}{2}\lambda\epsilon_{ii}\epsilon_{jj} + \mu\epsilon_{ij}\epsilon_{ij} + a_1\kappa_{iik}\kappa_{kjj} + a_2\kappa_{ijj}\kappa_{ikk} + a_3\kappa_{iik}\kappa_{jjk} \\ + a_4\kappa_{ijk}\kappa_{ijk} + a_5\kappa_{ijk}\kappa_{kji}$$

$$\tau_{ij} = \lambda\delta_{ij}\epsilon_{pp} + 2\mu\epsilon_{ij}$$

$$\mu_{ijk} = \frac{1}{2}a_1(\delta_{ij}\kappa_{kpp} + 2\delta_{jk}\kappa_{ppi} + \delta_{ik}\kappa_{jpp}) + 2a_2\delta_{jk}\kappa_{ipp} \\ + a_3(\delta_{ij}\kappa_{ppk} + \delta_{ik}\kappa_{ppj}) + 2a_4\kappa_{ijk} + a_5(\kappa_{kji} + \kappa_{jki})$$

- ▶ **Five** additional material parameters a_1, \dots, a_5
- ▶ $f_{ijkpq} = 0$ so τ_{ij} depends only on ϵ_{ij} and μ_{ijk} on κ_{ijk}

Equilibrium equation for displacements

- ▶ Introducing **two** lengths

$$l_1^2 = \frac{2(a_1 + a_2 + a_3 + a_4 + a_5)}{\lambda + 2\mu} > 0 \quad l_2^2 = \frac{a_3 + 2a_4 + a_5}{2\mu} > 0$$

- ▶ ... gives ($\nabla^2(\cdot) = (\cdot)_{,qq}$)

$$(\lambda + 2\mu)(1 - l_1^2 \nabla^2) u_{p,pk} - \mu(1 - l_2^2 \nabla^2)(u_{p,pk} - u_{k,pp}) + F_k = 0$$

- ▶ Simplified model: $a_1 = a_3 = a_5 = 0$,
 $a_2 = \lambda l^2/2$, $a_4 = \mu l^2$ so that $l_1 = l_2 = l$

$$(1 - l^2 \nabla^2)(\lambda u_{p,pk} + \mu(u_{p,pk} + u_{k,pp})) + F_k = 0$$

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Other gradient theories

Gradient plasticity and damage

Example problems

Numerical implementation

Strain-gradient plasticity and damage

- ▶ Second-gradient theory is a **general** framework
- ▶ Various classical models may be extended to the second-gradient case
- ▶ Gradient plasticity has been proposed and used to **regularise** problems of **strain localisation**
- ▶ Also some work on gradient damage theories

Theories with gradients of internal variables

- ▶ Plasticity and damage are theories with **internal variables**
- ▶ Introducing dependence on gradients of internal variables introduces an **internal length**
- ▶ The resulting models regularise localisation problems
- ▶ These “gradient plasticity” or “gradient damage” models are **not** second-gradient (of displacement) models

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Introduction

Gradient Elasticity

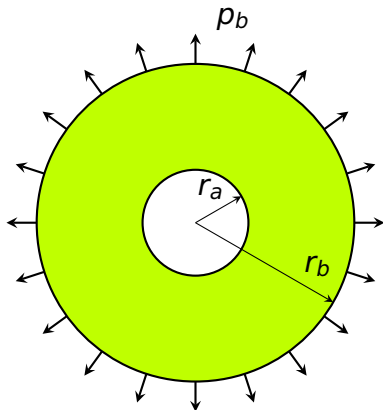
Other gradient theories

Example problems

- Thick hollow cylinder under external normal traction
- Infinite layer under shear
- Infinite layer with bolts
- Plane strain uniaxial loading
- Mode I crack

Numerical implementation

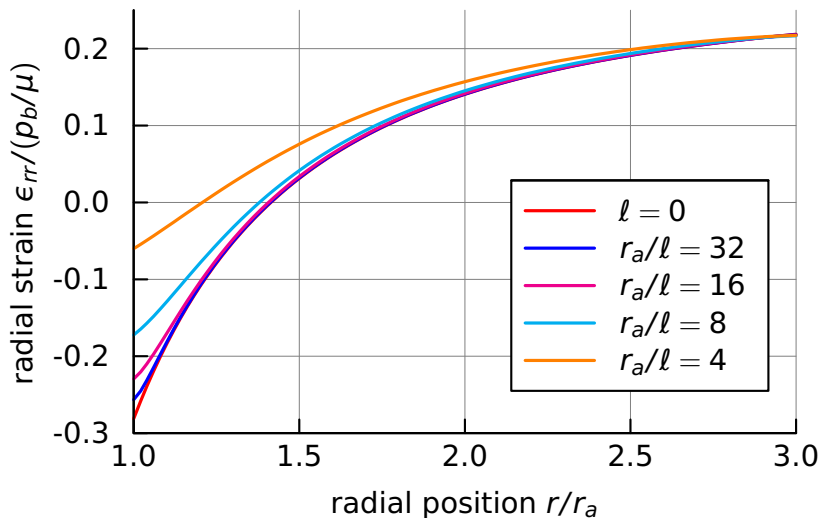
Thick hollow cylinder: description



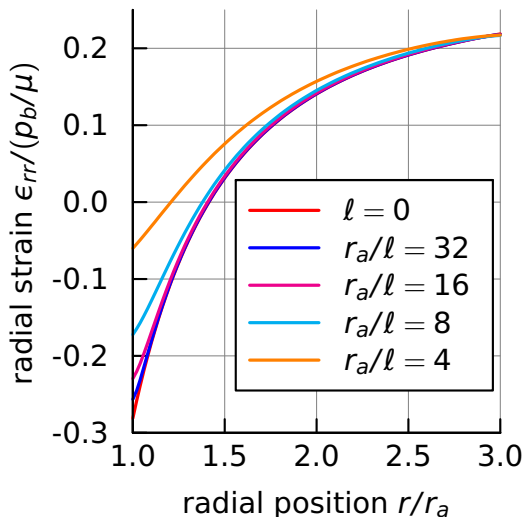
- ▶ **Simplified** isotropic linear second gradient elasticity, material length ℓ , plane strain (all examples)
- ▶ Only classical BCs
- ▶ $r_b/r_a = 3$, $\nu = 1/4$, different r_a/ℓ
- ▶ All results will be given in normalised (**dimensionless**) form

Thick hollow cylinder under external normal traction

Thick hollow cylinder: radial strain

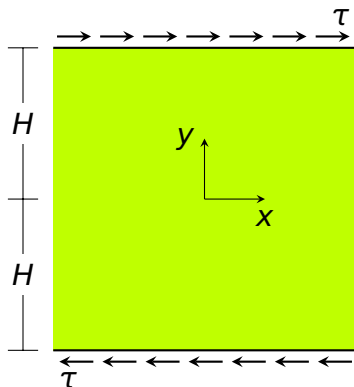


Thick hollow cylinder: radial strain



- ▶ Classical solution has strong strain gradient near inner boundary
- ▶ ... so difference of strain gradient solution larger
- ▶ Difference increases as r_a/l gets smaller (i.e. the hole gets smaller)

Shear layer: description



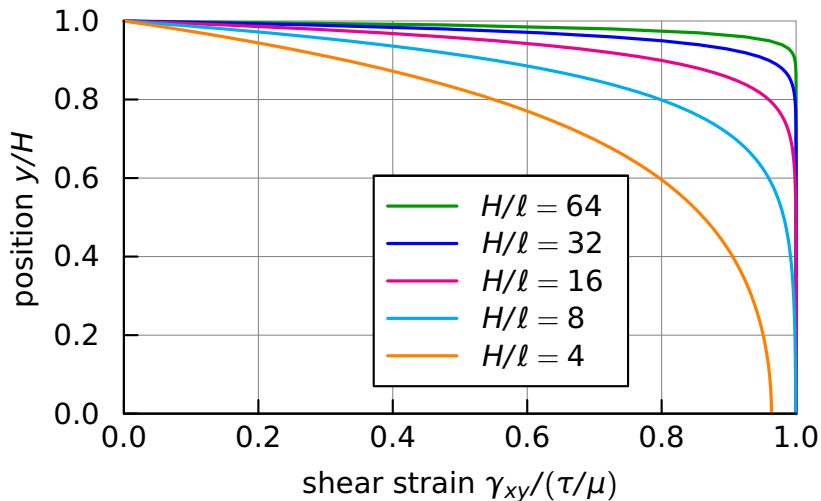
- ▶ Classical solution has **no** strain gradient
- ▶ We impose kinematic second-gradient BC

$$\partial u_x / \partial y = 0 \quad \text{for } y = \pm H$$

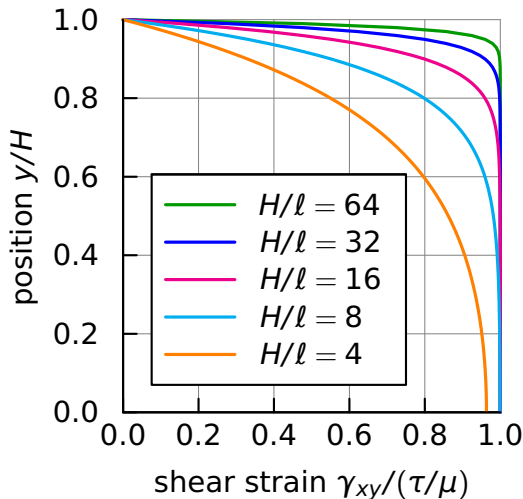
(“**rough boundary**” BC)

- ▶ Different H/ℓ
- ▶ Symmetry wrt x axis, consider only **upper half**

Shear layer: shear strain

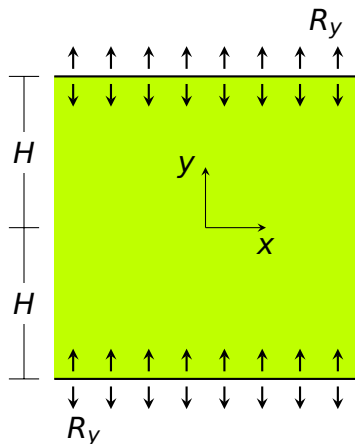


Shear layer: shear strain



- ▶ Classical solution cannot accommodate “rough boundary”
- ▶ Gradient solution creates **boundary layer**
- ▶ Size of boundary layer scales with l
- ▶ For l comparable to H , boundary layers merge

Bolted layer: description

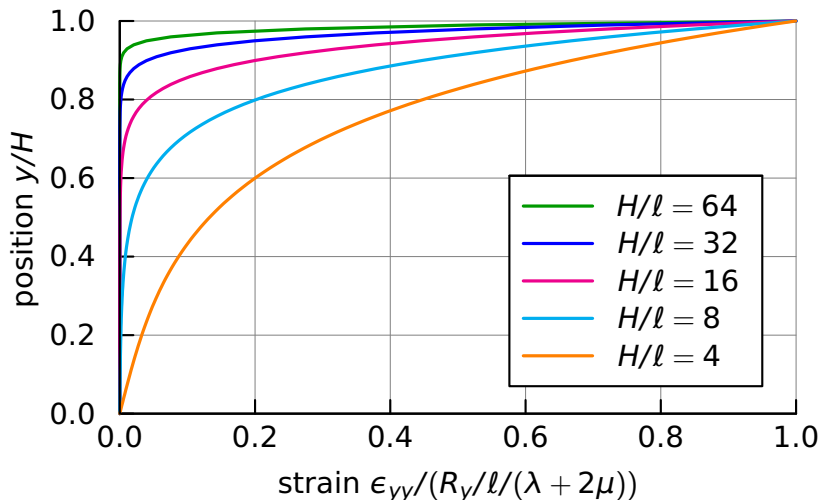


- ▶ We impose **static** second-gradient BC

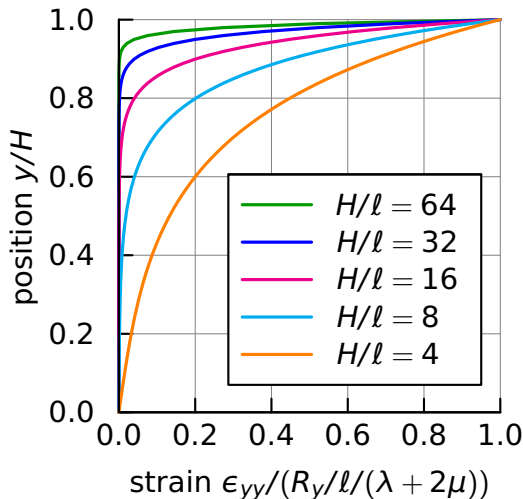
$$R_y \neq 0 \quad \text{for} \quad y = \pm H$$

- ▶ R_y is self-equilibrating
- ▶ Vardoulakis proposed $R_y < 0$ to simulate effect of rock bolts
- ▶ Symmetry wrt x axis, consider only **upper half**

Bolted layer: strain

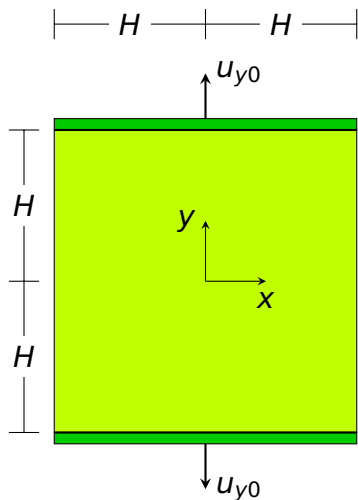


Bolted layer: strain



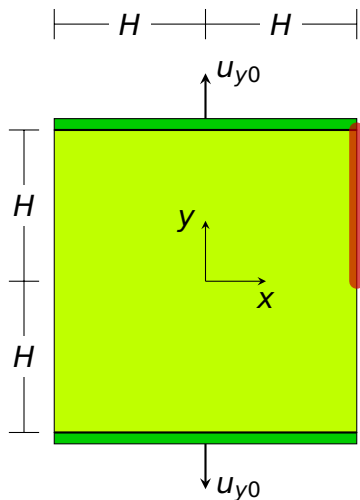
- ▶ Classical elasticity cannot accommodate BC for R_y
- ▶ **Surface** effect of R_y (self-equilibrating), creates boundary layer
- ▶ Size of boundary layer scales with l

Uniaxial loading: description



- ▶ Uniaxial loading in plane strain
- ▶ Classical solution has **no** strain gradient
- ▶ “Rough boundary” BC
$$\partial u_y / \partial y = 0 \quad \text{for } y = \pm H$$
- ▶ Different H/ℓ , $\nu = 0.35$

Uniaxial loading: description

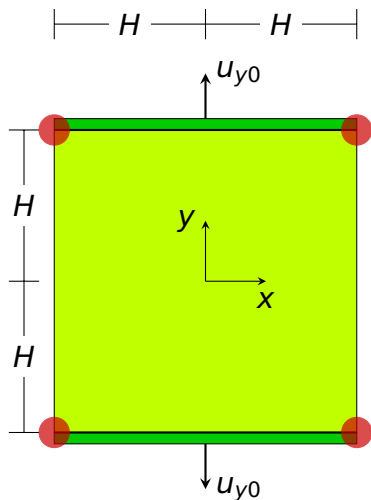


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$$\partial u_y / \partial y = 0 \quad \text{for } y = \pm H$$

- ▶ Different H/ℓ , $\nu = 0.35$
- ▶ Calculate displacements of **top half of right edge**

Uniaxial loading: description

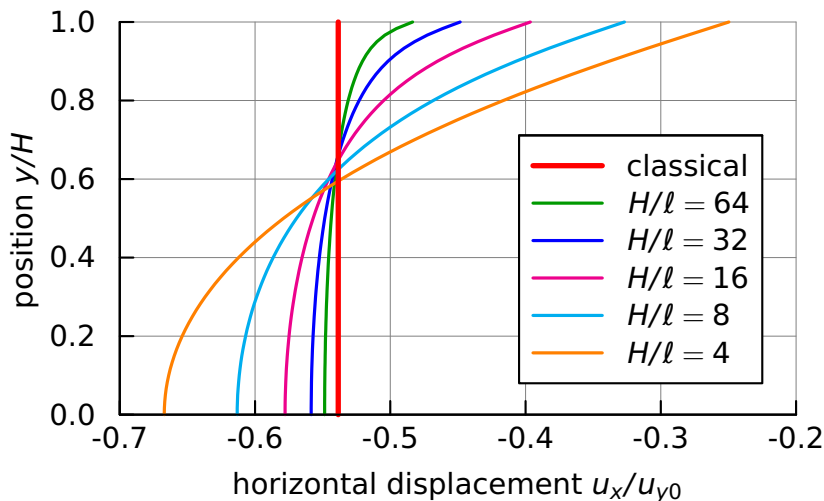


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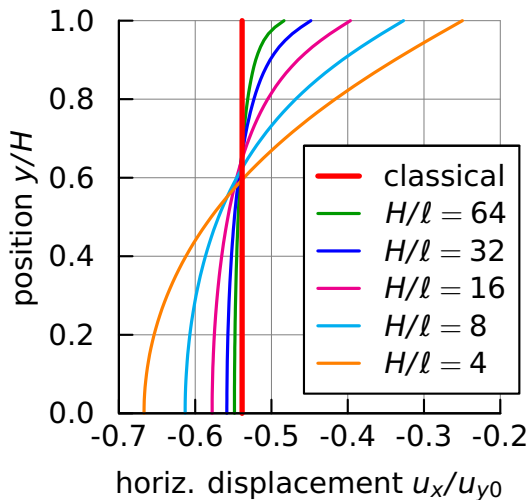
$$\partial u_y / \partial y = 0 \quad \text{for } y = \pm H$$

- ▶ Different H/ℓ , $\nu = 0.35$
- ▶ Calculate displacements of top half of right edge
- ▶ There are **corners** (edges in 3D)

Uniaxial loading: edge displacement

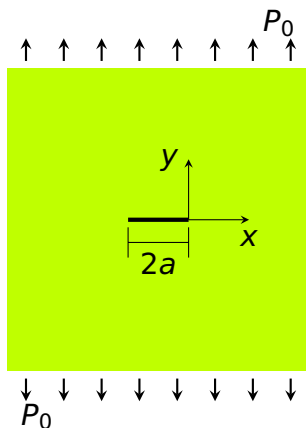


Uniaxial loading: edge displacement



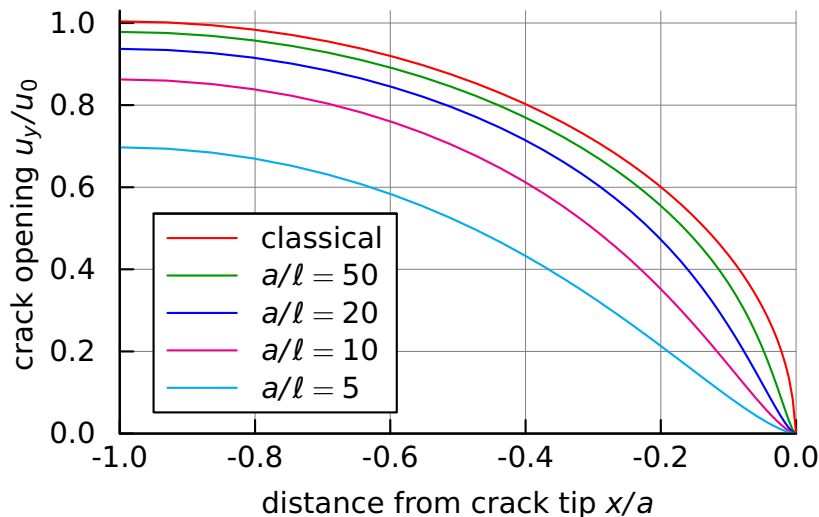
- ▶ Classical solution leaves sides straight (“uniform” solution)
- ▶ Gradient solution with “rough” BC distorts sides
- ▶ “Uniform” gradient solution impossible because of BC at corners $E_i = 0$
- ▶ Effect of corners (edges) even without edge tractions

Mode I crack: description

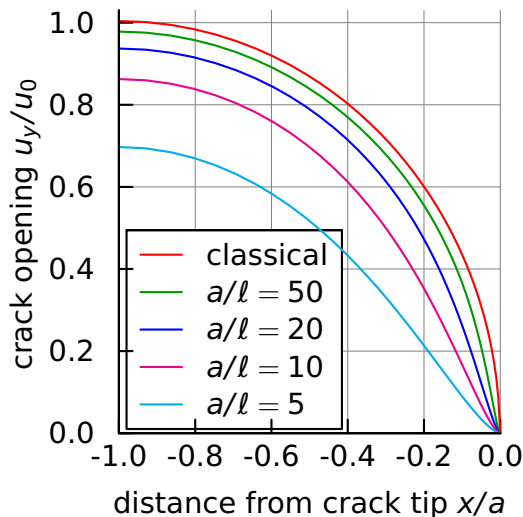


- ▶ Classical solution has **infinite** stress at crack tip
- ▶ Different a/l , $\nu = 0.2$
- ▶ Semi-analytical gradient solution exists
- ▶ **Numerical** results presented here

Mode I crack: crack opening

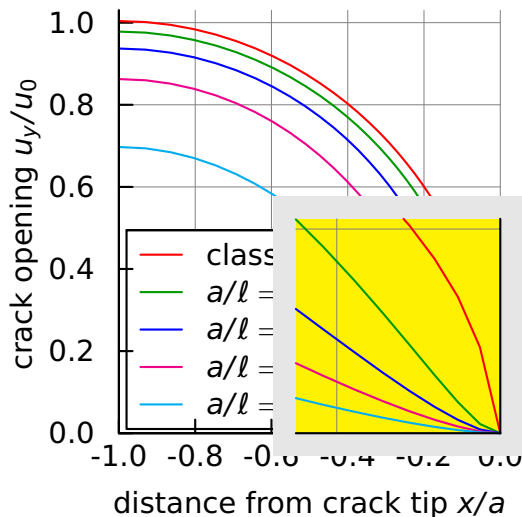


Mode I crack: crack opening



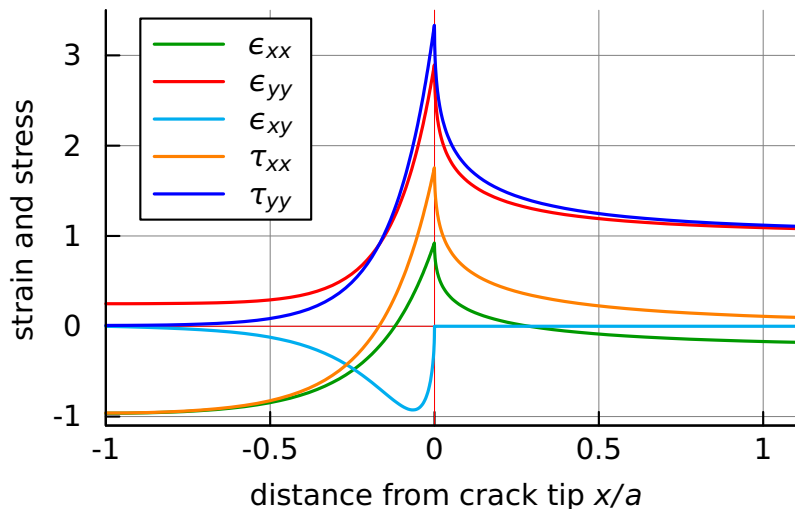
- ▶ Results for 1/4 of problem
- ▶ “Smaller” cracks become stiffer
- ▶ Cusped shape of crack ($\partial u_y / \partial x = 0$ at tip)
- ▶ Strains and stresses are important

Mode I crack: crack opening

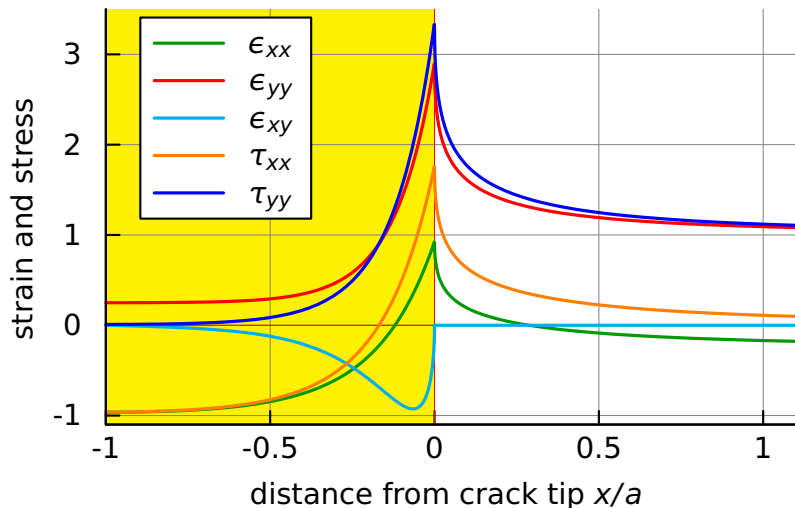


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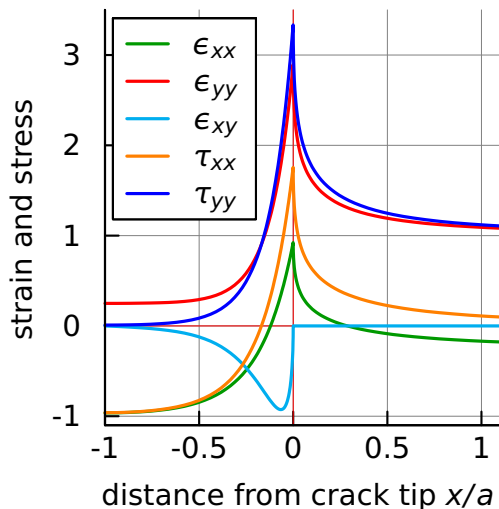
Mode I crack: strains and stresses



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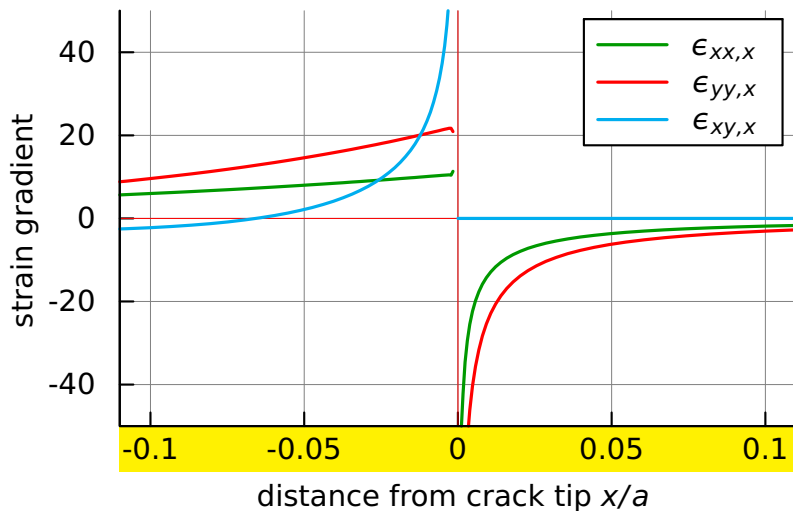


Mode I crack: strains and stresses

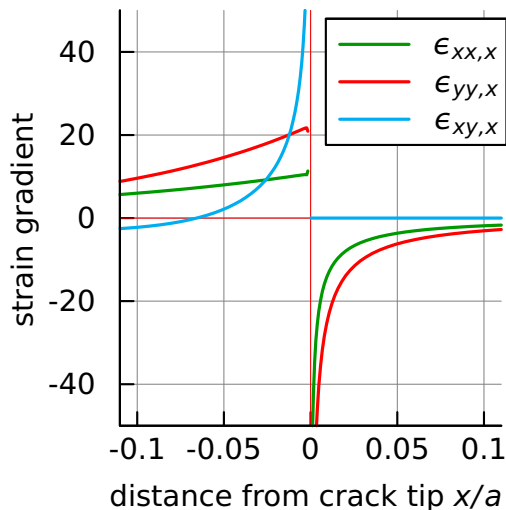


- ▶ Results for $a/l = 10$
- ▶ Strains and stresses at crack tip are
 - ▶ Finite
 - ▶ Continuous
 - ▶ Non-smooth
- ▶ We can use a maximum-stress criterion
- ▶ On crack face $\tau_{yy} \neq 0$
- ▶ Attention if other “stresses” are used

Mode I crack: strain gradients



Mode I crack: strain gradients



- ▶ Strain gradients and double-stresses at crack tip are **not continuous** and **infinite**
- ▶ Numerical solution affected by this

Outline

Introduction

Gradient Elasticity

Other gradient theories

Example problems

Numerical implementation

Basics of the Finite Element Method

C^1 discretisation of second-gradient models

Alternative numerical formulations

Very brief review of FEM

- ▶ Use **bold** notation
- ▶ Must calculate $\mathbf{u}(\mathbf{x})$ (displacement field)
- ▶ Interpolate using shape functions & DOFs
- ▶ $\mathbf{B}(\mathbf{x})$ contains first derivatives of $\mathbf{N}(\mathbf{x})$
- ▶ **Elasticity**: $\boldsymbol{\tau} = \mathbf{D}\boldsymbol{\epsilon}$
- ▶ Substituting in virtual work equation

$$\mathbf{u}(\mathbf{x}) = \mathbf{N}(\mathbf{x})\mathbf{u}^N$$

- ▶ Using matrix notation

$$\boldsymbol{\epsilon}^T \boldsymbol{\tau} = \tau_{ij} \epsilon_{ij}$$

$$\boldsymbol{\epsilon}(\mathbf{x}) = \mathbf{B}(\mathbf{x})\mathbf{u}^N$$

$$\left(\int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV \right) \mathbf{u}^N = \int_V \mathbf{N}^T \mathbf{F} dV + \int_S \mathbf{N}^T \mathbf{P} dS$$

or simply

$$\mathbf{K} \mathbf{u}^N = \mathbf{f}$$

The second-gradient case

- ▶ Still $\mathbf{K}\mathbf{u}^N = \mathbf{f}$ with $\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV$ and $\boldsymbol{\tau} = \mathbf{D}\boldsymbol{\epsilon}$, but:
 1. Vector of external actions

$$\mathbf{f} = \int_V \mathbf{N}^T \mathbf{F} dV + \int_S (\mathbf{N}^T \mathbf{P} + D(\mathbf{N}^T) \mathbf{R}) dS + \oint_C \mathbf{N}^T \mathbf{E} dC$$

2. Additional second-gradient terms in $\boldsymbol{\tau}$, $\boldsymbol{\epsilon}$, \mathbf{D}

$$\boldsymbol{\epsilon}^T \boldsymbol{\tau} = \tau_{ij} \epsilon_{ij} + \mu_{ijk} K_{ijk}$$

3. $\mathbf{B}(\mathbf{x})$ has both first and **second** derivatives of $\mathbf{N}(\mathbf{x})$
- ▶ Due to (3), **C^1 continuity** (continuous first derivative) required for interpolation of $\mathbf{u}(\mathbf{x})$
 - ▶ Usual finite elements **cannot** be used with second-gradient theories

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Usual finite elements **cannot** be used with 2nd-gradient theories

C^1 continuity of interpolation

- ▶ Polynomial interpolation within element is C^1
- ▶ Possible problem at **boundary** between elements
- ▶ 1D: single node [easy, use Hermite elements]
- ▶ 2D: edges & nodes [use plate-bending elements]
- ▶ 3D: faces, edges & nodes [more elements needed]

Note about gradients of internal variables

- ▶ The C^1 continuity requirement applies to second-gradient (strain-gradient) theories
- ▶ Plasticity/damage theories with gradients of internal variables **do not** generally require C^1 continuity
- ▶ Other difficulties are introduced
- ▶ We must calculate the gradient of a value evaluated at **discrete** (integration) points

Alternative numerical formulations

- ▶ C^1 requirement poses some limitations
 - ▶ Few elements available
 - ▶ Excessive continuity at nodes
- ▶ Alternative C^0 formulations are sought (simpler, “cheaper”, more flexible)
- ▶ Various **mixed** formulations have been proposed (discretise other fields besides displacement)
- ▶ Other possible methods
 - ▶ Meshless finite element methods
 - ▶ Boundary element method

Example of mixed formulation [1]

- ▶ Target is to avoid C^1 requirement. . .
- ▶ . . . created by second gradients (e.g. in \mathbf{B})
- ▶ Discretise **three** fields:
 1. $\mathbf{u} = \mathbf{N}\mathbf{u}^N$ displacements
 2. $\mathbf{v} = \mathbf{M}\mathbf{v}^N$ “relaxed” displacement gradients
 3. $\boldsymbol{\lambda} = \boldsymbol{\Lambda}\boldsymbol{\lambda}^N$ Lagrange multipliers
- ▶ Here \mathbf{v} is **not** defined as $\nabla\mathbf{u}$
- ▶ Constraint $\int_V \boldsymbol{\lambda} : (\nabla\mathbf{u} - \mathbf{v})dV = \mathbf{0}$ enforces $\nabla\mathbf{u} \approx \mathbf{v}$
- ▶ DOFs $\mathbf{u}^N, \mathbf{v}^N, \boldsymbol{\lambda}^N$ are not necessarily evaluated at the same positions (generally different interpolations $\mathbf{N}, \mathbf{M}, \boldsymbol{\Lambda}$ are used)

Example of mixed formulation [2]

- ▶ Introducing the “groupings”

$$\hat{\mathbf{u}} = \{\mathbf{u}, \mathbf{v}, \boldsymbol{\lambda}\} \quad \hat{\mathbf{u}}^N = \{\mathbf{u}^N, \mathbf{v}^N, \boldsymbol{\lambda}^N\}$$

$$\hat{\boldsymbol{\epsilon}} = \{\nabla \mathbf{u}, \nabla \mathbf{v}, \mathbf{v}, \boldsymbol{\lambda}\} \quad \hat{\boldsymbol{\tau}}^T = \{\boldsymbol{\tau} - \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\lambda}, \nabla \mathbf{u} - \mathbf{v}\}$$

$$\hat{\mathbf{N}} = \begin{bmatrix} \mathbf{N} & \cdot & \cdot \\ \cdot & \mathbf{M} & \cdot \\ \cdot & \cdot & \boldsymbol{\Lambda} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \nabla \mathbf{N} & \cdot & \cdot \\ \cdot & \nabla \mathbf{M} & \cdot \\ \cdot & \mathbf{M} & \cdot \\ \cdot & \cdot & \boldsymbol{\Lambda} \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{C} & \mathbf{F} & \cdot & -\mathbf{I} \\ \mathbf{F} & \mathbf{A} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \mathbf{I} \\ \mathbf{I} & \cdot & -\mathbf{I} & \cdot \end{bmatrix}$$

- ▶ We obtain the familiar relations

$$\hat{\mathbf{u}} = \hat{\mathbf{N}}\hat{\mathbf{u}}^N, \quad \hat{\boldsymbol{\epsilon}} = \mathbf{B}\hat{\mathbf{u}}^N, \quad \hat{\boldsymbol{\tau}} = \mathbf{D}\hat{\boldsymbol{\epsilon}}, \quad \hat{\boldsymbol{\epsilon}}^T \hat{\boldsymbol{\tau}} = \tau_{ij}\epsilon_{ij} + \mu_{ijk}K_{ijk}$$

- ▶ Now \mathbf{B} has up to **first** derivatives of shape functions
- ▶ \mathbf{D} non-symmetric, has some zero diagonal elements

Notes on C^1 vs mixed formulations

- ▶ Usual finite elements cannot be used
- ▶ All appropriate elements introduce **shortcomings** (but at least they work!)
- ▶ Mixed methods do offer greater flexibility (in the choice of elements)
- ▶ but also introduce one **additional** approximation
- ▶ **Computational cost** is an important factor
- ▶ C^1 elements generally have more DOFs (more expensive) but also higher interpolation (richer)
- ▶ When there is a C^1 interpolation, it is “optimum”
- ▶ Mixed elements discretise more fields than strictly necessary

Summary

- ▶ Continua with microstructure
- ▶ Second-gradient elasticity
- ▶ Other second-gradient models
- ▶ Example problems in second-gradient elasticity
- ▶ Numerical implementation issues