

03 Continua with microstructure: Cosserat Theory

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Introduction

- Classical continuum mechanics do not incorporate any intrinsic material length scale
- Classical continua consist of points having three translational dofs
 - Displacement in three directions u_x, u_y, u_z
- Material response to the displacement of its points
 - Symmetric stress tensor σ_{ij}
 - Transmission of loads is uniquely determined by a force vector, neglecting couples



- Such continua may be insufficient for description of certain physical phenomena
 - Real materials often have a number of important length scales, which should be included in a realistic model
 - Grains
 - Particles
 - Fibers
 - Cellular structures
 - Building blocks, etc.
- Non-classical behavior due to microstructure arises if the material is subjected to **non-homogeneous straining** and is mostly observed in regions of high strain gradients, e.g. at
 - Notches
 - Holes
 - Cracks



■ Typical components of

- Double stress μ_{ijk}
- Gradient deformations K_{ijk}

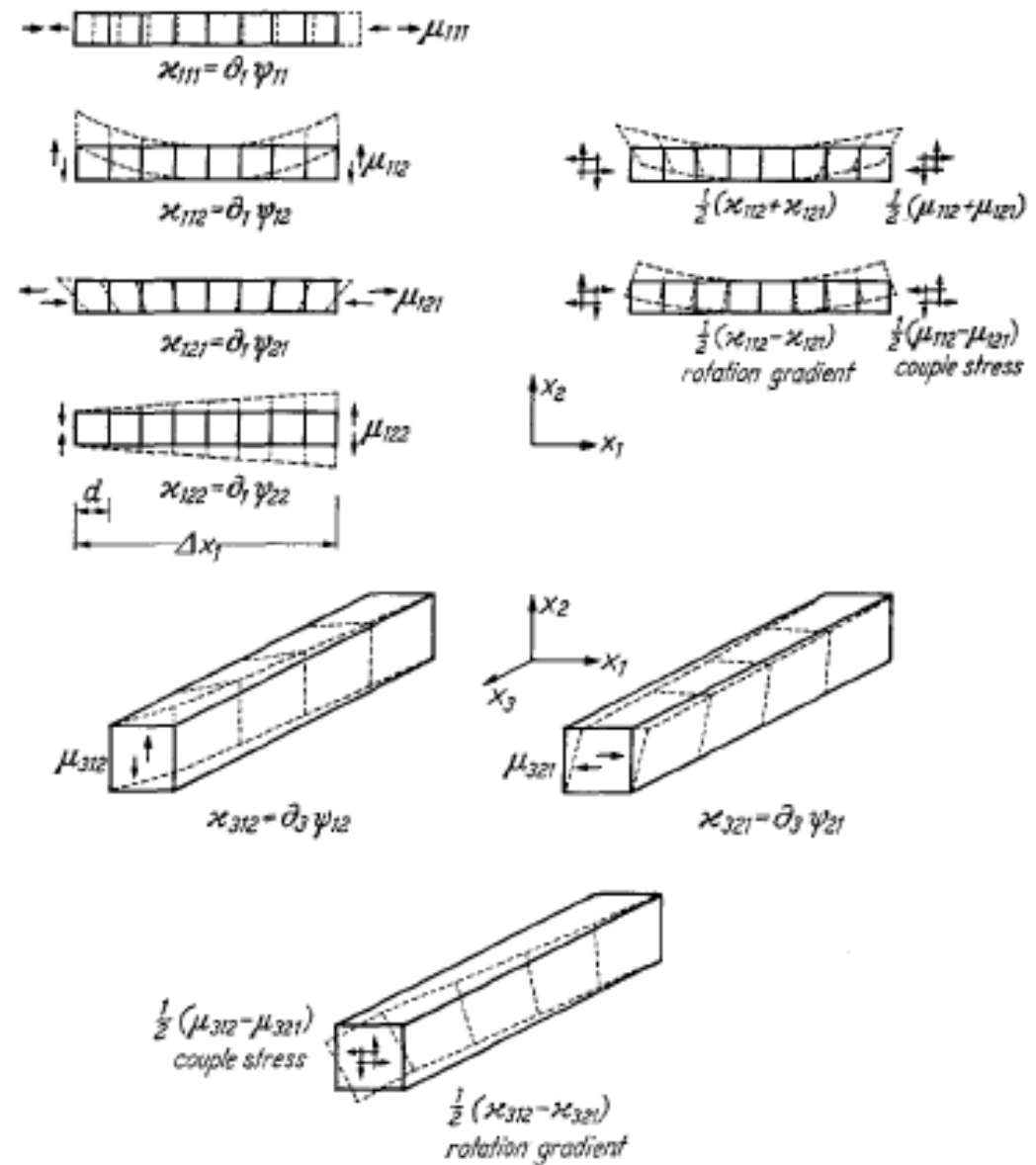


Fig. 2. Typical components of double stress μ_{ijk} and gradient of micro-deformation κ_{ijk}

- The Cosserat (or micropolar) continuum theory is one of the most prominent extended continuum theories
 - Seminal work of brothers Eugene and Francois Cosserat (1909)
 - Re-discovered in the 1950's (Mindlin, Gunther, etc.)
 - Milestone: Freudenstadt-IUTAM-Symposium on the Mechanics of Generalized Continua (Kröner 1968)
 - Introduction of micro-inertia for dynamic effects (Eringen 1970)
 - Historical account of the theory : Introduction of the CISM Lecture on Polar Continua by R Stojanović (1970)
 - Has been used 'extensively' since the 1980's



Cosserat continuum description

■ Kinematics

- Continuum of oriented *rigid particles*, called trièdres rigides (or rigid crosses), with 6 d.o.f.
 - 3 displacements u_i
 - 3 rotations ω_i^c (different from the rotational part ω_i of displacement gradient)

■ Statics

- 9 Force stresses σ_{ij} (force per unit area) associated with the displacements
- 9 Couple stresses μ_{ij} (torques per unit area) associated with the rotations

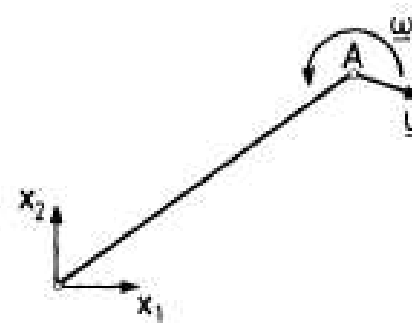
■ Constitutive relations (in simplest isotropic Cosserat elasticity)

- The 2 Lamé constants μ, λ of classical elasticity
- 1 additional elastic constant $R = \text{INTERNAL LENGTH}$ scale parameter
 - Relates to the microstructure
 - E.g. experimentally by the method of size effects for a particular problem

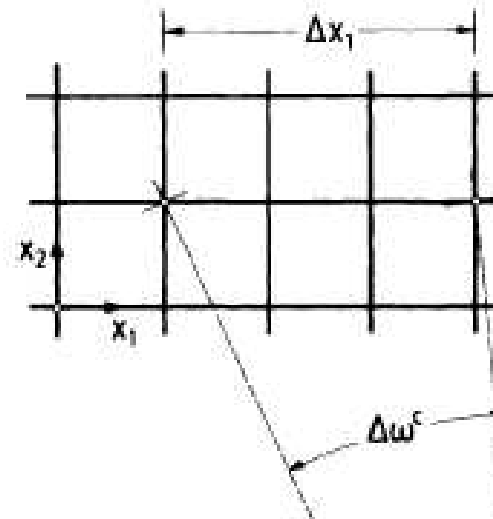


■ Visualization of Cosserat-continuum kinematics in 2-d

- (a) Displacement u_i and rotation ω_3^c of a rigid cross
- (b) Relative rotation $\Delta\omega_3^c$ of two neighboring rigid crosses (curvature)

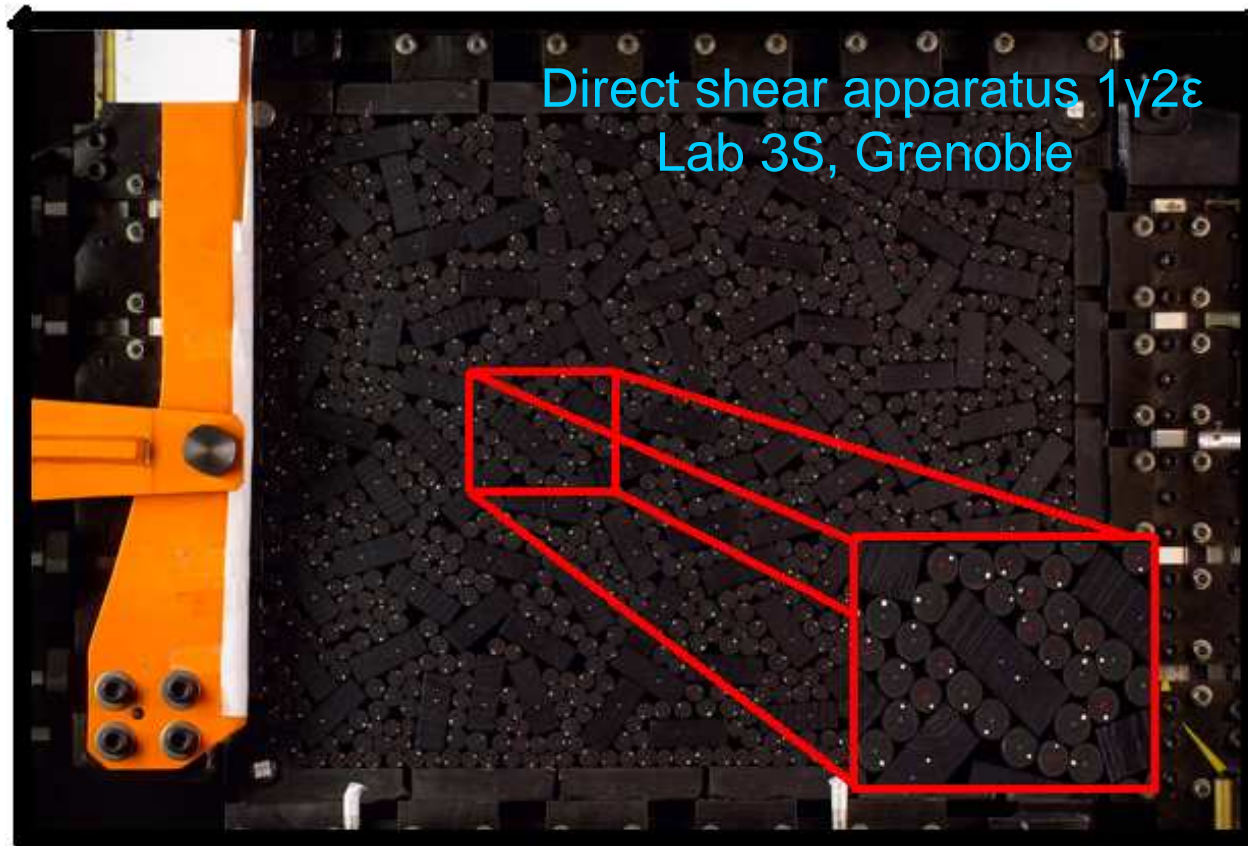


(a)



(b)

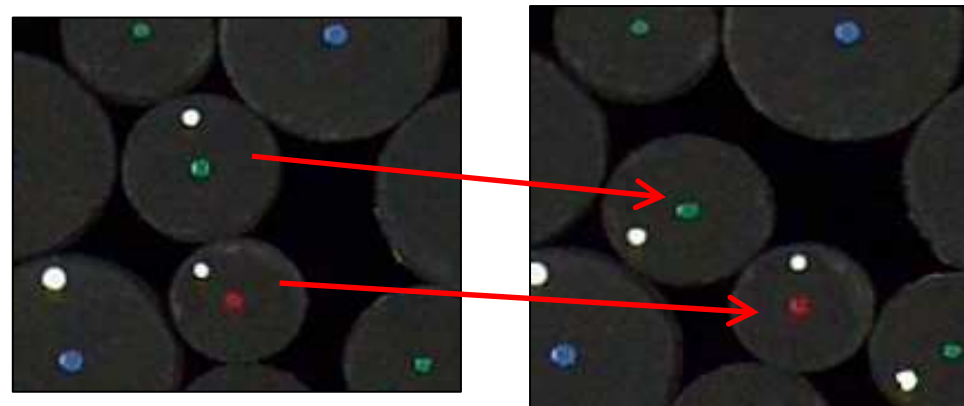
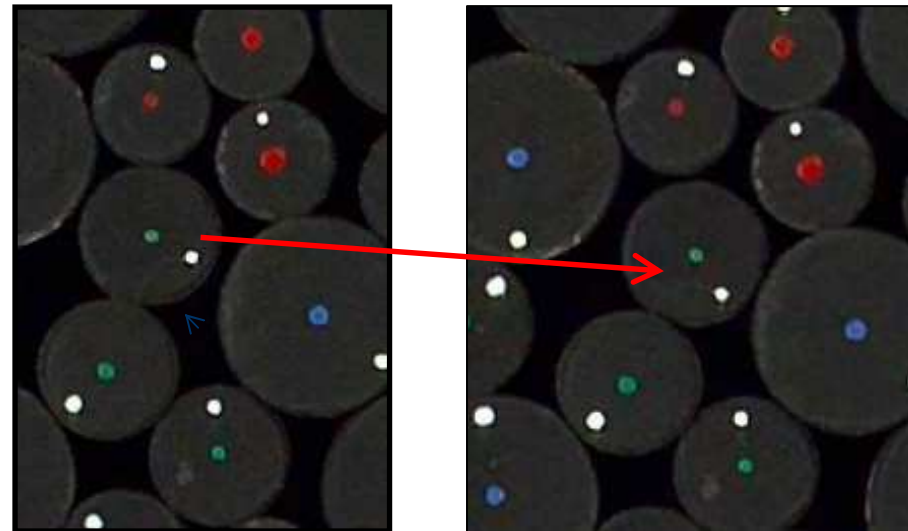
- Continuous medium consisting of a collection of particles that behave like rigid bodies
 - E.g. Assembly of rods and bricks



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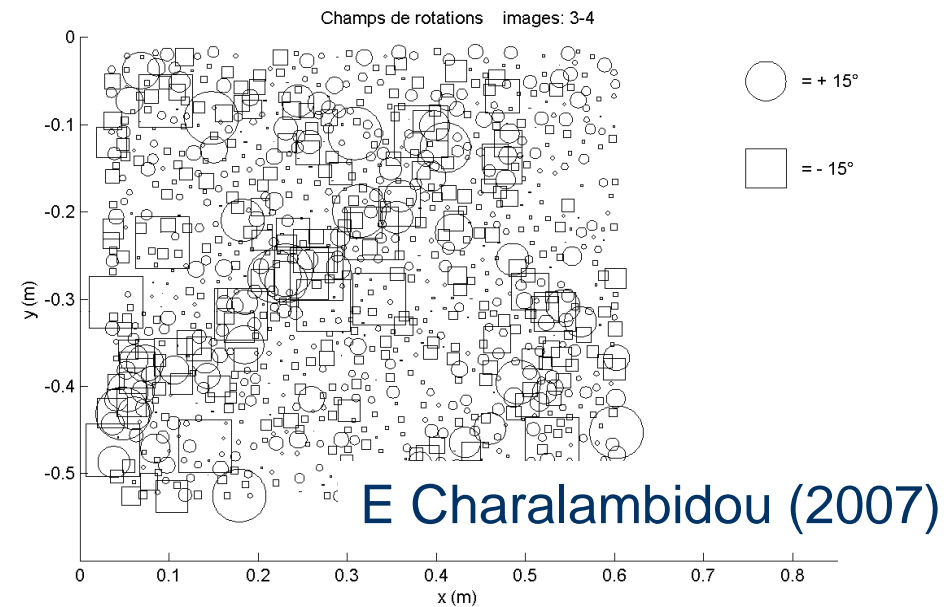
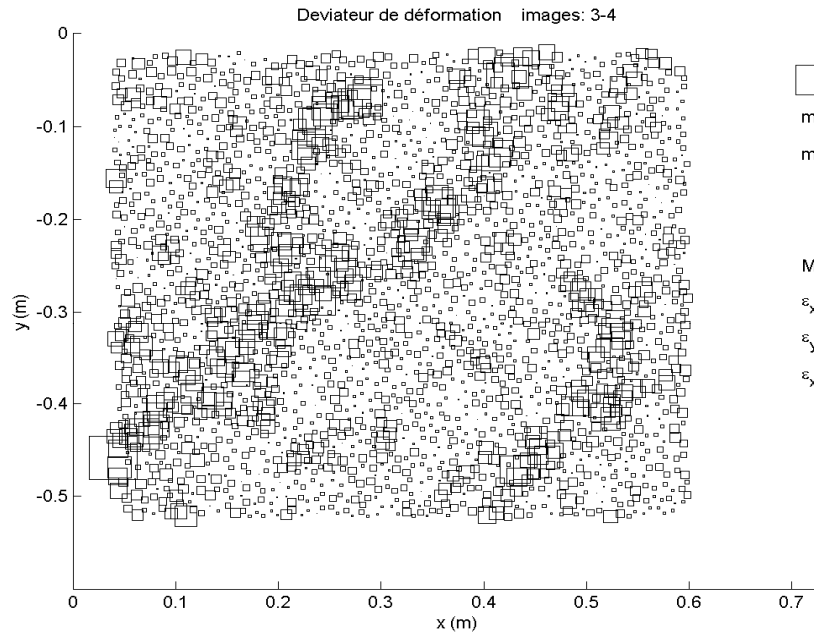


- Rotation of individual particles differs from that of their neighborhood



E Charalambidou (2007)

Large rotations in areas of non-uniform deformation, e.g. in shear bands



■ Deformation field

■ Rotation field



Brick structure

H.-B. Mühlhaus: Application of Cosserat theory in nume

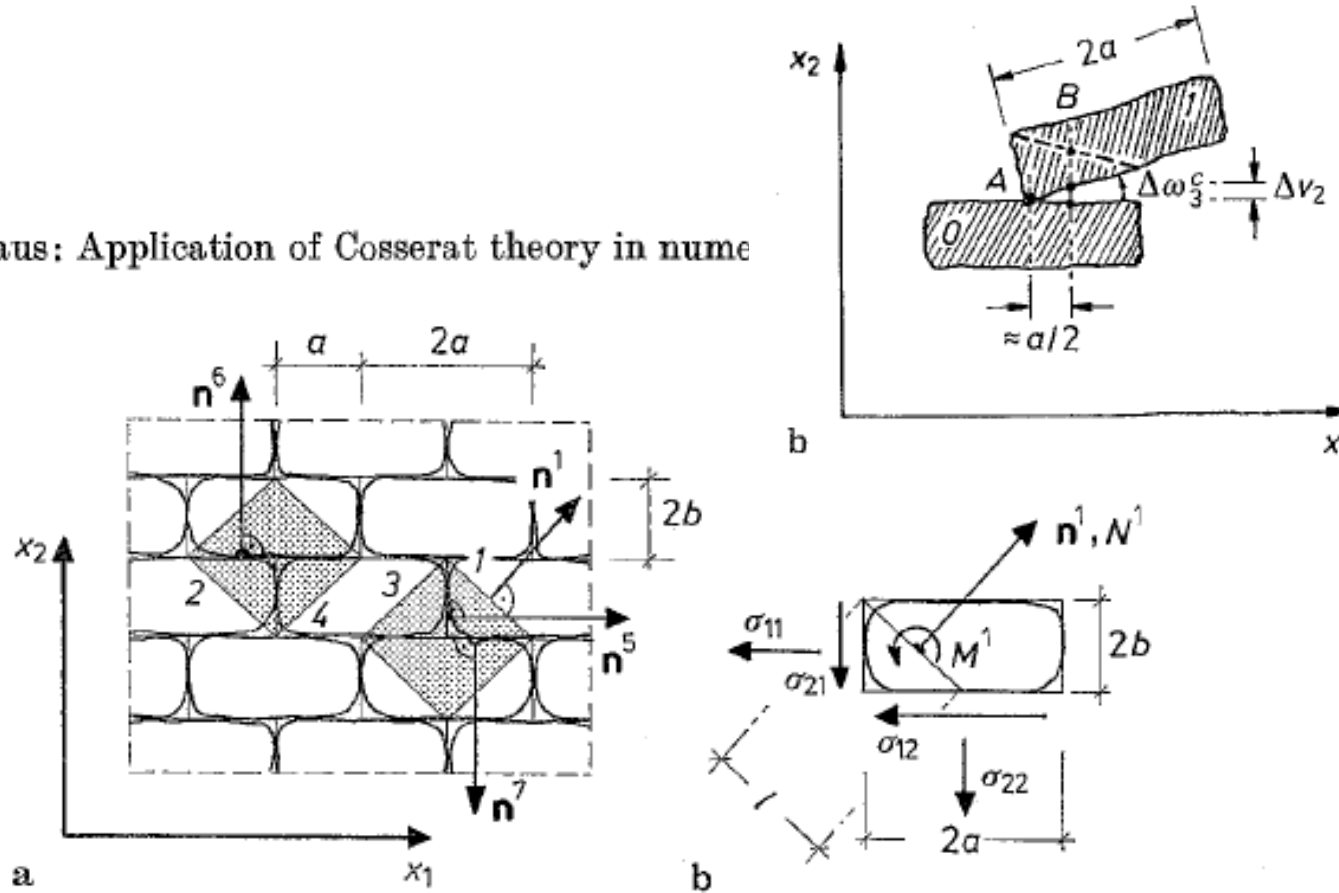
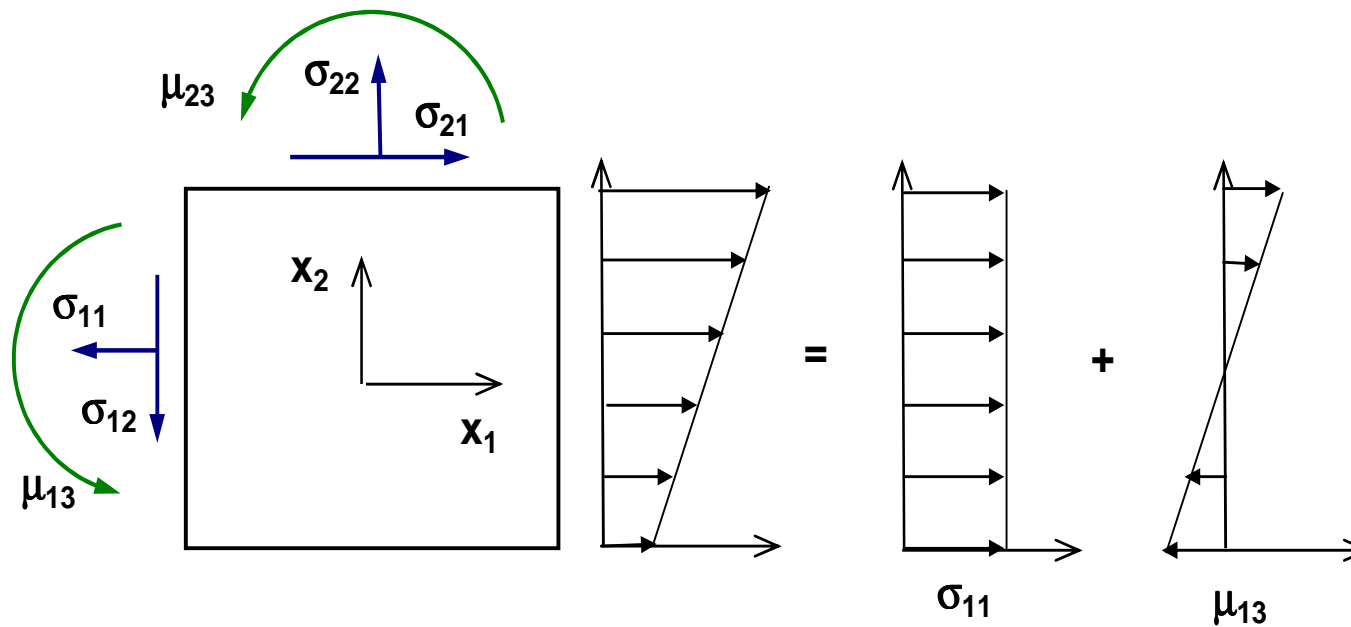


Fig. 2a and b. Geometry of the block structure (periodic elements are shaded)

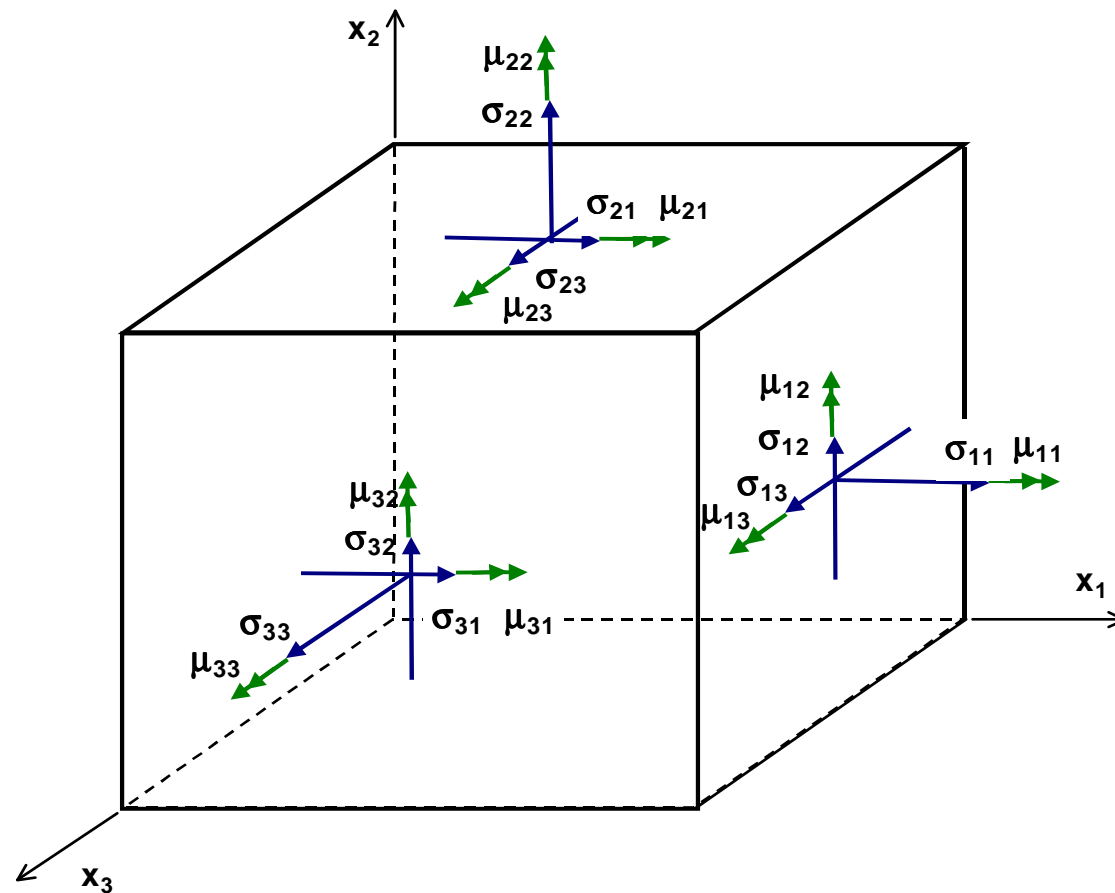
■ Statics of a **2-d** Cosserat stress element

- 4 (force) stresses σ_{ij}
- 2 couple stresses μ_{ij}



■ Statics of a 3-d Cosserat stress element

- 9 (force) stresses σ_{ij}
- 9 couple stresses μ_{ij}
 - Bending
 - Torsional
- Sign convention



■ Derivation of equilibrium equations

- (a) Force equilibrium
- (b) Moment equilibrium

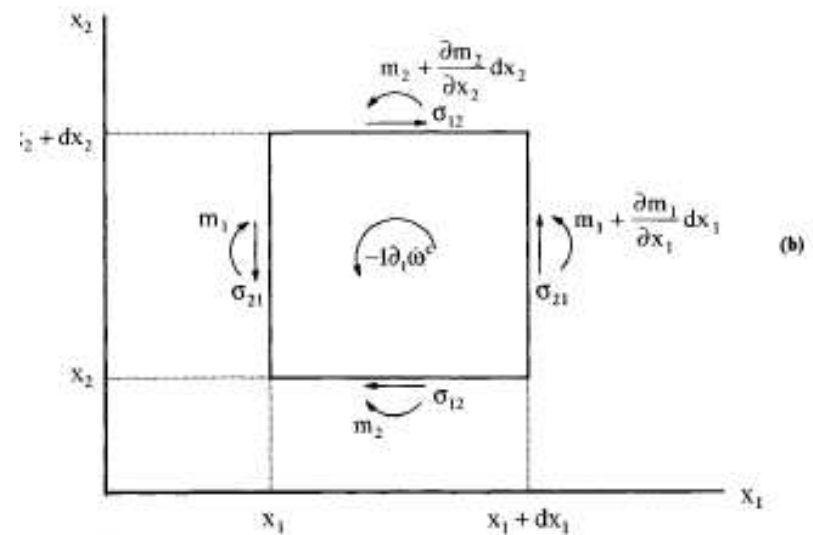
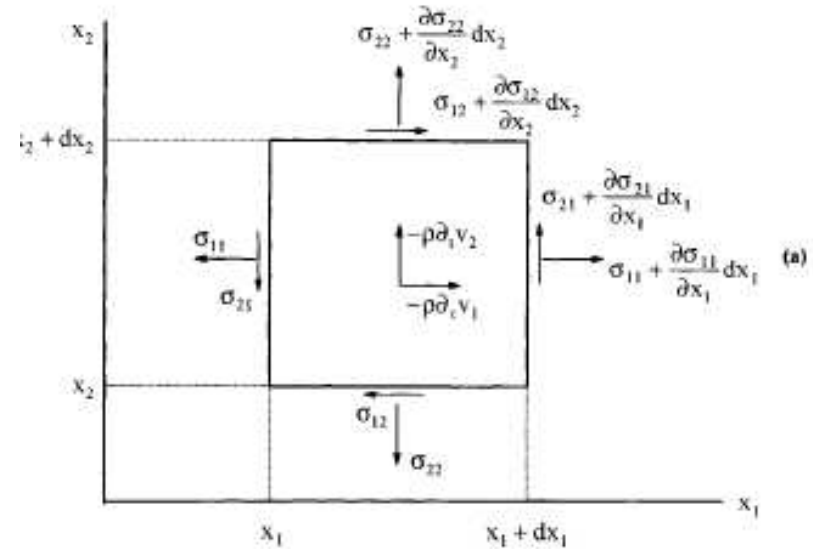
$$\sigma_{ji,j} = 0 \quad \text{in } V$$

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0$$

$$\mu_{ji,j} + e_{ijk} \sigma_{jk} = 0 \quad \text{in } V$$

$$\frac{\partial \mu_{xz}}{\partial x} + \frac{\partial \mu_{yz}}{\partial y} + \sigma_{xy} - \sigma_{yx} = 0$$



Non-symmetry of stress tensor

- 1 of the essential features of Cosserat or micropolar continua
 - Modified equation for the balance of angular momentum
- All theories in which the stress tensor is not symmetric can be regarded as polar-continua
- Typically predict a size-effect, meaning that smaller samples of the same material behave relatively stiffer than larger samples
 - An experimental fact completely neglected in the classical approach
 - It implies that the additional parameter in the Cosserat model defines a length-scale present in the material



Example

Circular hole in a field of simple tension

- Experiments show that "Brittle fracture and onset of static yielding in the presence of stress concentration occur at higher loads than expected on the basis of stress concentration factors calculated from the theory of elasticity" (Mindlin 1963)
- Kirsch solution => Stress concentration = 3
- Couple stress solution

$$\frac{\sigma_m}{p} = \frac{3+F}{1+F} \quad F = \frac{8(1-\nu)}{4 + \frac{a^2}{R^2} + \frac{2a}{R} \frac{K_0(a/R)}{K_1(a/R)}}$$

$$\frac{\sigma_{m|a/R=\infty}}{p} = 3$$

$$\frac{\sigma_{m|a/R=3}}{p} = \frac{3+0.44(1-\nu)}{1+0.44(1-\nu)} \approx 2.4 - 2.6$$



Example

Circular hole in a field of simple tension

- Stress concentration factor with increasing hole size (Mindlin 1993)



Characteristic differences between classical and Cosserat continuum elasticity (Lakes 2010)

Classical continuum

- In bending and torsion of circular cylindrical bars the rigidity is proportional to the fourth power of the diameter
- Wave speed of plane shear and dilatational waves in an unbounded medium is independent of frequency
- No length scale; hence stress concentration factors for holes or inclusions under a uniform stress field depend only on the shape of the inhomogeneity and not its size

Cosserat continuum

- Size-effect in bending and torsion of circular cylinders/square bars
 - Slender cylinders appear stiffer than expected classically
 - Experimentally one uses size effects to determine the internal length
- Speed of shear wave depends on frequency
 - A new kind of wave associated with the micro-rotation is predicted.
- Stress concentration factor for a circular hole is smaller than the classical value (Mindlin 1963)
 - Small holes exhibit less stress concentration than larger ones
 - This gives rise to enhanced toughness



Some problems

- Micropolar model has been around for several decades but yet there is no compelling evidence for the use of such a model
- Material moduli which characterize the model have NOT been measured directly
- Problems with regard to the prescription of boundary conditions for the microrotations

I.A. Kunin (1982, 1983)



Fields of application

(Neff 2010)

- *Regularization technique for post-failure computations*
 - E.g. in elasto-plasticity where shear-banding occurs
 - Geomechanics has been an area where this technique has been used with success
 - Cosserat internal length can be taken as numerical tuning value
 - Cosserat model conveys the shear band a definite width instead of an ill-defined shear band width in the non-polar case causing mesh-dependent results in FEM-simulations (Mühlhaus and Vardoulakis 1987)
- *Replacement medium for a granular assembly*
 - Large amount of work has been dedicated to this problem
 - Relevant values for material parameters not really clear since in these applications typically a multifold of physical processes takes place at the same time, like elasticity, plasticity, viscous relaxation etc.
 - Experimentally validated that particle rotations are an important factor in the development of shear bands in granular materials
 - Similar considerations apply to masonry and blocky structures

2



- *Homogenization of structural elements, discrete mass-spring systems and periodic microstructure*
 - Relate the beam structure geometry to various material moduli
 - Notably, the smallest distance between grid points can be related to the internal length scale in the Cosserat model

- *Model for the prediction of size-effects in many structures (e.g. boreholes, e.g. Papanastasiou 1988) and foam type materials like bones or cellular materials (e.g. Lakes 1998, 2010).*
 - Cosserat parameters determined by size experiments

- *Three-dimensional model for rigorous derivation of shell and plate models*
 - Rigorous derivations for otherwise ad hoc mechanical plate models like the well-known Reissner-Mindlin membrane-bending plate



■ *Dynamic problems*

- Differences between classical linear elasticity and experiment appear particularly in dynamic problems involving elastic vibrations of high frequencies and short wavelength
- The reason lies in the microstructure of the material
- Linear dynamic Cosserat model predicts dispersion
 - Wave speed depends on the wave frequency
 - Impossible in classical linear elasticity, but an experimental fact
 - New rotational waves that have not been observed yet experimentally

■ *Micropolar fluid flow*

- For modeling turbulence and augmenting the Navier-Stokes equations



Formulation of Cosserat continuum

■ Deformation field

■ Macro-strain ϵ_{ij}

$$\epsilon_{ij} = (u_{i,j} + u_{j,i})/2$$

■ Macro-rotation ω_{ij}

$$\omega_i = e_{ijk} (u_{j,k} - u_{k,j})/4$$

■ Relative strain ϵ_{ij}
(difference between macro-displacement gradient and micro-rotation)

$$\epsilon_{ij} = u_{j,i} - e_{ijk} \omega_k^c$$

■ Cosserat rotation gradient (curvature) κ_{ij}

$$\kappa_{ij} = \omega_{j,i}^c$$

$$\epsilon_{ij} = \epsilon_{ij} + e_{ijk} (\omega_k - \omega_k^c)$$

relative rotation of a material point
wrt. rotation of its neighborhood

For = 0 => Couple stress theory
(or Restricted Cosserat)

e_{ijk} = alternating tensor



Principle of virtual work

■ Introduce

- Non-symmetric stress σ_{ij}
- Non-symmetric couple stress μ_{ij}

- dual in energy to the deformation measures ε_{ij} and κ_{ij} , respectively

$$\int_V (\sigma_{ij} \delta \varepsilon_{ij} + \mu_{ij} \delta \kappa_{ij}) dV = \int_{S_\sigma} (t_i \delta u_i + m_i \delta \omega_i^c) dS$$

t_i = surface traction

m_i = surface couple traction



Variational equilibrium equation

$$\int_V \sigma_{ji,j} \delta u_i dV + \int_V (\mu_{ji,j} + e_{ijk} \sigma_{jk}) \delta \omega_i^c dV =$$

$$= \int_{S_\sigma} (\sigma_{ji} n_j - t_i) \delta u_i dS + \int_{S_\sigma} (\mu_{ji} n_j - m_i) \delta \omega_i^c dS$$

■ 6 Equilibrium equations

$$\sigma_{ji,j} = 0 \quad \text{in } V$$

$$\mu_{ji,j} + e_{ijk} \sigma_{jk} = 0 \quad \text{in } V$$

■ 6 Traction boundary conditions

$$\sigma_{ji} n_j = t_i \quad \text{in } S_\sigma$$

$$\mu_{ji} n_j = m_i \quad \text{in } S_\sigma$$

■ 6 Displacement b.c.

$$u_i = U_i \quad \text{in } S_u$$

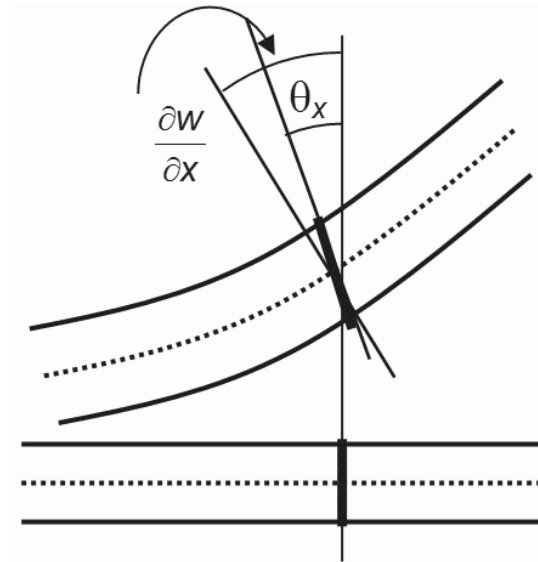
$$\omega_i^c = \Omega_i^c \quad \text{in } S_u$$



Example of b.v. problem

Pure bending of a Cosserat-elastic beam

- Timoshenko beam = 1-d Cosserat beam



$$EI \frac{d^4 w}{dx^4} = q(x) - \frac{EI}{kAG} \frac{d^2 q}{dx^2}$$

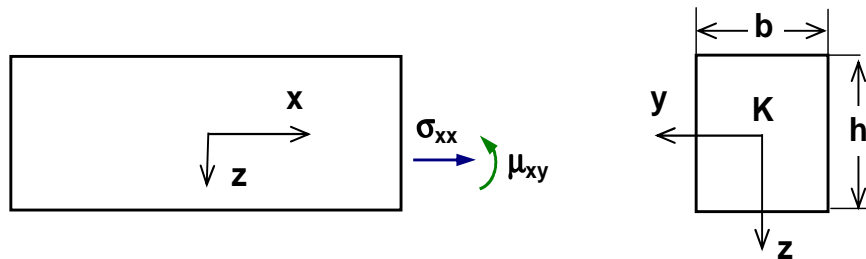
Deformation of a **Timoshenko** beam
The normal rotates by an amount not equal to dw / dx

Vardoulakis 2009

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Pure bending of a Cosserat-elastic beam



- Non trivial stresses are the axial stress σ_{xx} and the couple stress μ_{xy} and their corresponding strain ε_{xx} and curvature κ_{xy}
- The geometry of deformation gives
 - $\rho =$ radius of curvature
- Compatibility conditions

$$\varepsilon_{xx} = z/\rho$$

$$\implies \kappa_{xy} = 1/\rho$$

$$\frac{\partial \varepsilon_{xx}}{\partial z} - \kappa_{xy} = 0$$

■ Elasticity relations

$$\sigma_{xx} = E\varepsilon_{xx} = \frac{Ez}{\rho}$$

■ Satisfies equilibrium equations

$$\mu_{xy} = 1.6GR^2 \kappa_{xy} = \frac{1.6GR^2}{\rho}$$

$$\frac{\partial \sigma_{xx}}{\partial x} = 0$$

$$\frac{\partial \mu_{xy}}{\partial x} = 0$$

■ Total bending moment

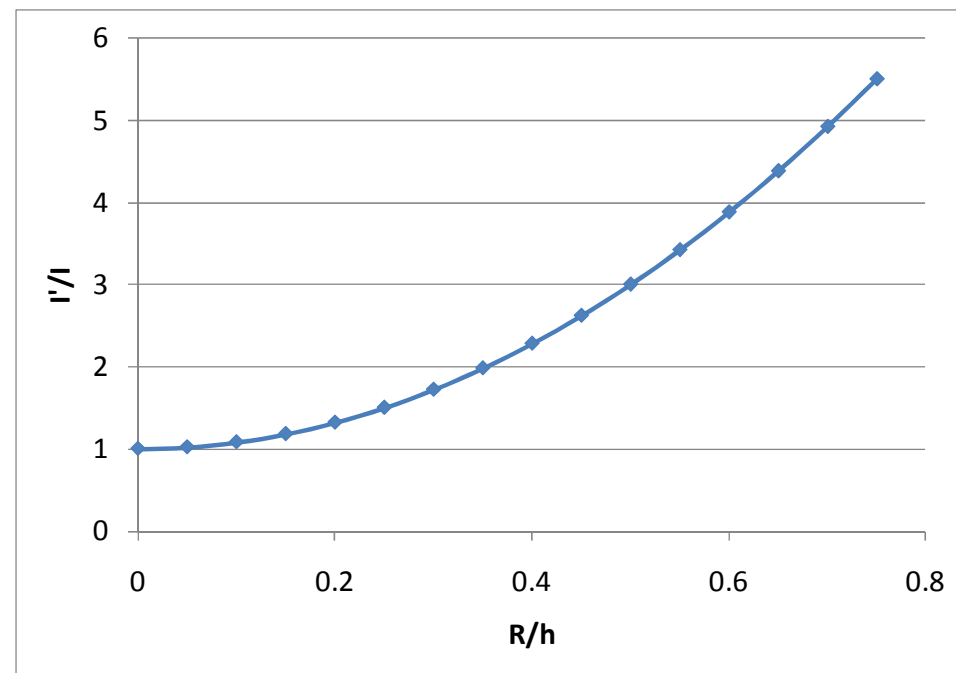
$$\begin{aligned} M &= \int_{-h/2}^{h/2} \sigma_{xx} z b dz + \int_{-h/2}^{h/2} \mu_{xy} b dz = \frac{E}{\rho} b \int_{-h/2}^{h/2} z^2 dz + \frac{1.6GR^2}{\rho} b \int_{-h/2}^{h/2} dz = \\ &= \frac{E}{\rho} \frac{bh^3}{12} + \frac{1.6GR^2}{\rho} bh = \frac{E}{\rho} \left(I + \frac{0.8R^2}{1+\nu} bh \right) = \frac{EI}{\rho} \left[1 + \frac{9.6}{1+\nu} \left(\frac{R}{h} \right)^2 \right] \end{aligned}$$



Cosserat beam is stiffer

- I' = equivalent moment of inertia of a Cosserat beam
- $I' \rightarrow I$ for $R/h \rightarrow 0$
- The smaller the structure the larger the effect

$$M = \frac{EI'}{\rho} \quad , \quad I' = I \left[1 + \frac{9.6}{1+\nu} \left(\frac{R}{h} \right)^2 \right]$$



Conclusions

- Mechanics of Cosserat continua can be useful in the modeling of heterogeneous materials when the size of the heterogeneities (internal length) and the size of the structure (external length) are of the same order of magnitude
- Useful in modeling size effects and as a regularization method for numerical computations where classical continua analysis breaks down
- However, the limited range of applicability, the calibration of the internal length and the boundary conditions for the micro-rotations are curtailing their use

